Review of homogeneous recurrences

$$a_n = \sum_{i<n} k_i a_i$$

Example: $F_n = F_{n-1} + F_{n-2}$ (Fibonacci) ✓

$a_n = 2a_{n-1} + 1$ (Tower of Hanoi) ✗

$R_n = R_{n-1} + n$ (Regions with n lines) ✗

Transform to homogeneous:

eliminate this.

$$a_n = 2a_{n-1} + 1$$

$$a_{n-1} = 2a_{n-2} + 1$$

$$a_n - a_{n-1} = 2a_{n-1} - 2a_{n-2}$$

$$a_n = 3a_{n-1} - 2a_{n-2} \qquad \text{(homogeneous)}$$

The Form $a_n = Aa_{n-1} + Ba_{n-2}$

largest index is n

smallest index is $n-2$

$$n - (n-2) = 2$$

$$\Downarrow$$

form a degree 2 (quadratic) equation

$$a_n = A a_{n-1} + B a_{n-2}$$

$$x^2 = Ax + B \quad (\text{characteristic Eq.})$$

<u>Example 1:</u>

Tower of Hanoi: $a_n = 3a_{n-2} - 2a_{n-2} \quad (A=3,\ B=-2)$

$$x^2 = 3x - 2 \quad \begin{matrix} \nearrow p = 1 \\ \searrow q = 2 \end{matrix}$$

$$a_n = c_1 p^n + c_2 q^n = c_1 + c_2 2^n$$

$$\left.\begin{aligned} a_0 &= c_1 + c_2 2^0 = c_1 + c_2 = 0 \\ a_1 &= c_1 + c_2 2^1 = c_1 + 2c_2 = 1 \end{aligned}\right| \Longrightarrow c_1 = -1,\ c_2 = 1$$

$$a_n = 2^n - 1 .$$

Example 3. $a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3}$ (Exploding Tower)

Characteristic Eq: $x^3 = 2x^2 + x - 2 \Rightarrow x^3 - 2x^2 - x + 2 = 0$

$p \quad q \quad r$

How to solve? Guess! $x=1$ works.

$$a_n = c_1 p^n + c_2 q^n + c_3 r^n$$

Rewrite:

$+2$

$$(x-1)(x^2 + \boxed{}\,x - 2) = 0$$

$+x^3$

$\boxed{} = -1$ since $ax^2 - x^2 = -2x^2$

Solve $x^2 - x - 2 = 0$

$\nearrow 2$

$\searrow -1$

$$a_n = c_1(1)^n + c_2\, 2^n + c_3\, (-1)^n$$

$$a_n = c_1 + c_2\, 2^n + c_3\, (-1)^n$$

$$\begin{aligned}
a_0 &= c_1 + c_2 + c_3 = 0 \\
a_1 &= c_1 + 2c_2 - c_3 = 0 \\
a_2 &= c_1 + 4c_2 + c_3 = 1
\end{aligned}$$

$$2c_1 + 3c_2 = 0$$

$$2c_1 + 6c_2 = 1$$

$$2c_1 = -1$$

$$c_1 = -\frac{1}{2}$$

$$c_1 = -\frac{1}{2} \qquad c_2 = \frac{1}{3} \qquad c_3 = \frac{1}{6}$$

$$a_n = -\frac{1}{2} + \frac{1}{3}\, 2^n + \frac{1}{6}(-1)^n$$

$$a_n = \frac{2^{n+1} - 3 + (-1)^n}{6}$$

$$a_n = A a_{n-1} + B a_{n-2} + C a_{n-3}$$

When solving for $x^3 - Ax^2 - Bx - C = 0 \to p,\ q,\ r$

p, q, r all different:

$$c_1 p^n + c_2 q^n + c_3 r^n$$

$p = q \neq r$:

$$c_1 p^n + c_2 n p^n + c_3 r^n$$

$p = q = r$:

$$c_1 p^n + c_2 n p^n + c_3 n^2 p^n$$

Example 2

$$R_n = R_{n-1} + n$$

$$\ominus \quad R_{n-1} = R_{n-2} + (n-1)$$

$$R_n - R_{n-1} = R_{n-1} - R_{n-2} + 1$$

$$R_n = 2R_{n-1} - R_{n-2} + 1$$ (still not homogeneous)

$$\ominus \quad R_{n-1} = 2R_{n-2} - R_{n-3} + 1$$

$$R_n - R_{n-1} = 2R_{n-1} - R_{n-2} - 2R_{n-2} + R_{n-3}$$

$$R_n = 3R_{n-1} - 3R_{n-2} + R_{n-3}$$

(Equation will have degree 3)

$$x^3 = 3x^2 - 3x + 1$$

$$x^3 - 3x^2 + 3x - 1 = 0 \Rightarrow (x-1)^3 = 0$$

$$(x-1)(x-1)(x-1) = 0$$

Continue: 3 solutions that are the same: $p=q=r=1$

$$a_n = c_1 p^n + c_2 n p^n + c_3 n^2 p^n = c_1 + c_2 n + c_3 n^2$$

$$a_0 = c_1 = 1$$

$$\left.\begin{array}{l} a_1 = c_1 + c_2 + c_3 = 2 \\ a_2 = c_1 + 2c_2 + 4c_3 = 4 \end{array}\right| \Rightarrow -c_1 + 2c_3 = 0 \Rightarrow c_3 = \frac{c_1}{2} = \frac{1}{2}$$

$$c_2 = \frac{1}{2}$$

$$a_n = 1 + \frac{1}{2} n + \frac{1}{2} n^2 = \frac{n(n+1)}{2} + 1 .$$

Another example:

$$a_n = 4a_{n-1} - 3a_{n-2} + 2^n$$

$$a_{n-1} = 4a_{n-2} \quad 3a_{n-3} + 2^{n-1}$$

$$2a_{n-1} = 8a_{n-2} - 6a_{n-3} + 2^n$$

$$a_n - 2a_{n-1} = 4a_{n-1} \underbrace{- 3a_{n-2} - 8a_{n-2}} + 6a_{n-3}$$

$$a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$$

$$x^3 = 6x^2 - 11x + 6$$

$$x^3 - 6x^2 + 11x - 6 = 0$$

Guess $p=1$ is a solution.

$$(x-1)(x^2 + \boxed{}\, x + 6) = 0$$

$$\square\, x^2 - x^2 = -6x^2$$

$$\square = -5$$

$$(x-1)(x^2-5x+6)=0$$

$$p=1 \qquad q=2 \qquad r=3$$

$$a_n = c_1 p^n + c_2 q^n + c_3 r^n$$

$$= c_1 + c_2 2^n + c_3 3^n$$

$$a_0 = c_1 + c_2 + c_3 = ? \quad (\text{given } a_0)$$

$$a_1 = c_1 + 2c_2 + 3c_3 = ? \quad \text{"} \quad a_1$$

$$a_2 = c_1 + 4c_2 + 9c_3 = ? \quad \text{"} \quad a_2$$