

## Review of homogeneous recurrences

$$a_n = \sum_{i < n} k_i a_i$$

Example:  $F_n = F_{n-1} + F_{n-2}$  (Fibonacci) ✓

$a_n = 2a_{n-1} + 1$  (Tower of Hanoi) X

$R_n = R_{n-1} + n$  (Regions with n lines) X

Transform to homogeneous:

$$a_n = 2a_{n-1} + 1$$

eliminate this.

$$\underline{a_{n-1} = 2a_{n-2} + 1}$$

$$a_n - a_{n-1} = 2a_{n-1} - 2a_{n-2}$$

$$a_n = 3a_{n-1} - 2a_{n-2} \quad (\text{homogeneous})$$

The Form  $a_n = Aa_{n-1} + Ba_{n-2}$

largest index is  $n$  ↗      ↙ smallest index is  $n-2$

$$n - (n-2) = 2$$

↓  
form a degree 2 (quadratic) equation

$$a_n = Aa_{n-1} + Ba_{n-2}$$

$$x^2 = Ax + B \quad (\text{characteristic Eq.})$$

Example 1 :

Tower of Hanoi :  $a_n = 3a_{n-1} - 2a_{n-2}$  ( $A=3$ ,  $B=-2$ )

$$x^2 = 3x - 2$$

$\nearrow p=1$   
 $\searrow q=2$

$$a_n = c_1 p^n + c_2 q^n = c_1 + c_2 z^n$$

$$a_0 = c_1 + c_2 2^0 = c_1 + c_2 = 0$$

$$a_1 = c_1 + c_2 2^1 = c_1 + 2c_2 = 1$$

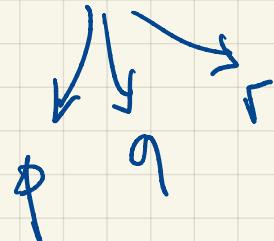
$$\Rightarrow c_1 = -1, c_2 = 1$$

$$a_n = 2^n - 1 .$$

Example 3.  $a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3}$  (Exploding Tower)

Characteristic Eq:  $x^3 = 2x^2 + x - 2 \Rightarrow x^3 - 2x^2 - x + 2 = 0$

How to solve? Guess!  $x=1$  works.



Rewrite:

$$(x-1)(x^2 + \boxed{\square}x - 2) = 0$$

+ 2  
↓      ↓  
(x-1) (x<sup>2</sup> + □x - 2) = 0  
↑      ↑  
+ x<sup>3</sup>

$$a_n = C_1 P^n + C_2 q^n + C_3 r^n$$

□ = -1 since  $0x^2 - x^2 = -2x^2$

Solve  $x^2 - x - 2 = 0$

$$\begin{array}{c} 2 \\ \swarrow \quad \searrow \\ -1 \end{array}$$

$$a_n = c_1(1)^n + c_2 2^n + c_3 (-1)^n$$

$$a_n = c_1 + c_2 2^n + c_3 (-1)^n$$

$$a_0 = c_1 + c_2 + c_3 = 0 \quad \begin{cases} 2c_1 + 3c_2 = 0 \\ 2c_1 + 6c_2 = 1 \end{cases}$$

$$a_1 = c_1 + 2c_2 - c_3 = 0 \quad \begin{cases} 2c_1 = -1 \\ c_1 = -\frac{1}{2} \end{cases}$$

$$a_2 = c_1 + 4c_2 + c_3 = 1 \quad \begin{cases} c_1 = -\frac{1}{2} \\ c_2 = \frac{1}{3} \\ c_3 = \frac{1}{6} \end{cases}$$

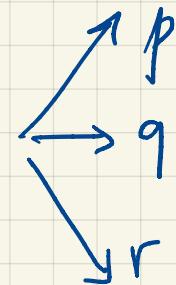
$$c_1 = -\frac{1}{2} \quad c_2 = \frac{1}{3} \quad c_3 = \frac{1}{6}$$

$$a_n = -\frac{1}{2} + \frac{1}{3} 2^n + \frac{1}{6} (-1)^n$$

$$a_n = \frac{2^{n+1} - 3 + (-1)^n}{6}$$

$$a_n = Aa_{n-1} + Ba_{n-2} + Ca_{n-3}$$

When solving for  $x^3 - Ax^2 - Bx - C = 0$



$p, q, r$  all different:

$$c_1 p^n + c_2 q^n + c_3 r^n$$

$p = q \neq r$ :

$$\underbrace{c_1 p^n + c_2 n p^n}_{\text{red}} + c_3 r^n$$

$p = q = r$ :

$$\underbrace{c_1 p^n + c_2 n p^n + c_3 n^2 p^n}_{\text{red}}$$

## Example 2

$$R_n = R_{n-1} + n$$

$$\textcircled{-} \quad R_{n-1} = R_{n-2} + (n-1)$$

$$\underline{R_n - R_{n-1} = R_{n-1} - R_{n-2} + 1}$$

$$R_n = 2R_{n-1} - R_{n-2} + 1 \quad (\text{still not homogeneous})$$

$$\textcircled{-} \quad \underline{R_{n-1} = 2R_{n-2} - R_{n-3} + 1}$$

$$\underline{R_n - R_{n-1} = 2R_{n-1} - R_{n-2} - 2R_{n-2} + R_{n-3}}$$

$$R_n = 3R_{n-1} - 3R_{n-2} + R_{n-3}$$

(Equation will have degree 3)

$$x^3 = 3x^2 - 3x + 1$$

$$x^3 - 3x^2 + 3x - 1 = 0 \Rightarrow (x-1)^3 = 0$$

$$(x-1)(x-1)(x-1) = 0$$

Continue: 3 solutions that are the same:  $p=q=r=1$

$$a_n = C_1 p^n + C_2 n p^n + C_3 n^2 p^n = C_1 + C_2 n + C_3 n^2$$

$$a_0 = C_1 = 1$$

$$a_1 = C_1 + C_2 + C_3 = 2$$

$$a_2 = C_1 + 2C_2 + 4C_3 = 4 \quad \Rightarrow \quad -C_1 + 2C_3 = 0 \Rightarrow C_3 = \frac{C_1}{2} = \frac{1}{2}$$

$$C_2 = \frac{1}{2}$$

$$a_n = 1 + \frac{1}{2}n + \frac{1}{2}n^2 = \frac{n(n+1)}{2} + 1 .$$

Another example:

$$a_n = 4a_{n-1} - 3a_{n-2} + 2^n$$

$$a_{n-1} = 4a_{n-2} - 3a_{n-3} + 2^{n-1}$$

$$2a_{n-1} = 8a_{n-2} - 6a_{n-3} + 2^n$$

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$$a_n - 2a_{n-1} = 4a_{n-1} - \underbrace{3a_{n-2}}_{-8a_{n-2}} - \underbrace{8a_{n-2}}_{+6a_{n-3}} + 6a_{n-3}$$

$$a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$$

$$x^3 - 6x^2 - 11x + 6$$

$$x^3 - 6x^2 + 11x - 6 = 0$$

Guess  $p=1$  is a solution.

$$(x-1)(x^2 + \boxed{\phantom{0}}x + 6) = 0$$

$$\boxed{x^2} - x^2 = -6x^2$$

$$\boxed{ } = -5$$

$$(x-1)(x^2 - 5x + 6) = 0$$

$$\begin{matrix} \downarrow \\ p=1 \end{matrix} \quad \begin{matrix} \downarrow \\ q=2 \end{matrix} \quad \begin{matrix} \downarrow \\ r=3 \end{matrix}$$

$$a_n = C_1 P^n + C_2 q^n + C_3 r^n$$

$$= C_1 + C_2 2^n + C_3 3^n$$

$$a_0 = C_1 + C_2 + C_3 = ? \text{ (given } a_0 \text{)}$$

$$a_1 = C_1 + 2C_2 + 3C_3 = ? \text{ " } a_1$$

$$a_2 = C_1 + 4C_2 + 9C_3 = ? \text{ " } a_2$$