Discrete Math Test 2 Spring 2025

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DO NOT WRITE HERE

Name:

P1:

EmplID:

P2:

Recitation Instructor (circle one):

P3:

Shayan (shokri) on Mon   Anthony on Tue/Wed   Taha on Thu

P4:

Note: Write clearly in the dedicated space,
and use ink to answer the questions, not a pencil.

Do not turn this page before it's time to start, but while you wait, make sure you wrote your name, and you may draw something in the space below.

Problem 1: Where triangular numbers show up...
Let $T_n = \sum_{i=1}^{n} i = 1 + 2 + \ldots + n = n(n+1)/2$, for all integers $n \geq 0$.

(a) (1 point) The Fibonacci sequence starts with $F_0 = 0$ and $F_1 = 1$, where $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. We observe that both F_0 and F_1 satisfy the following property:

$$F_0 = T_0 \quad \text{and} \quad F_1 = T_1$$

Prove that **there exists** at least one more value of n such that $F_n = T_n$.

(b) (2 points) The following fact will be useful for this problem: Given an integer n, $n(n+1)/(n-1)$ is never an integer when $n > 3$.

For integer $n > 3$, prove **by contradiction** that there is no $x, y \in \{1, 2, 3, \ldots, n\}$ such that $x + y$ is the average of all other integers (the integers in the set excluding x and y).

Hint: **1)** Express the statement that must mark the beginning of your proof-by-contradiction, then **2)** write that statement mathematically (what is the average of all numbers not including x and y in terms of n, T_n, x, and y?), and finally **3)** solve for $x + y$ and see where that takes you... (Each part is worth some points, so make sure that you make it clear.)

Problem 2: Colorful sets
Call a set $\{x, y\} \subset \mathbb{N}$ *colorful* iff x and y have different parity (that means one is even and one is odd).

(a) (2 points) Prove the following statement using its **contrapositive**:

$$x + y + xy \text{ is even } \Rightarrow \{x, y\} \text{ is not colorful}$$

(b) (1 point) Show by a **counter example** that the above implication (stated in part (a)) cannot be reversed.

(c) (1 point) Let's generalize the notion of a colorful set. A subset of $\mathbb{N}$ is colorful if some of its elements are even and some are odd. Let C be the set of all colorful sets. Which of the following statements do you agree with? Choose one of the four and justify your answer:

- The set C is a subset of $\mathcal{P}(\mathbb{N})$, the power set of $\mathbb{N}$. So C is "smaller" than $P(\mathbb{N})$ and, therefore, there is no bijection between C and $\mathcal{P}(\mathbb{N})$. Since $\mathcal{P}(\mathbb{N})$ is uncountable, C must be countable because the next infinity bigger than the cardinality of $\mathbb{N}$ is that of $\mathcal{P}(\mathbb{N})$.

- Consider the set of all subsets of $\mathbb{N}$ that contain only even integers, call it E. Define the set O in the same way for odd. Now, $\mathcal{P}(\mathbb{N}) = C \cup E \cup O$. If C is countable, so are E and O, and $\mathcal{P}(\mathbb{N})$ would be the union of three countable sets, and thus countable. But that's not true. So C is uncountable.

- With $\mathcal{P}(\mathbb{N})$, E, and O defined above, observe that E is uncountable because there is a bijection between $\mathcal{P}(\mathbb{N})$ and E (multiply elements of a set in $\mathcal{P}(\mathbb{N})$ by 2). Since E is uncountable, so is O. At some level, we may think of C as $E \times O$ (by listing the even elements of a set first followed by the odd ones), so C is uncountable since it's the product of uncountable sets.

- The set C must be countable, for one thing, it's name is given by the first letter of "countable". In addition, $\mathbb{N}$ itself is in C, and if C is uncountable, that makes $\mathbb{N}$ uncountable as well, which is not true.

Problem 3: Sunflowers

Consider a collection of two or more sets $S_1, S_2, \ldots, S_n$, each of which is a subset of $\mathbb{N}$.

Definition: The collection is called a *sunflower* of size n iff there exists a set T called *kernel*, such that:

$$\forall\ 1 \leq i \neq j \leq n, S_i \cap S_j = T$$

In other words, the intersection of every pair of sets in the sunflower is the same. For example, $\Big\{\{1,2,3,4\},\{1,3,5,7\},\{1,3,9,27\}\Big\}$ is a sunflower of size 3 with $S_1 = \{1,2,3,4\}$, $S_2 = \{1,3,5,7\}$, and $S_3 = \{1,3,9,27\}$ and has kernel $T = S_1 \cap S_2 = S_1 \cap S_3 = S_2 \cap S_3 = \{1,3\}$.

The following questions must be answered in general, and not in reference to the above example.

(a) (1 point) In your opinion, why is the term *sunflower* used in the above definition?

(b) (1 point) Given any set S_i in a sunflower with kernel T, explain clearly why $S_i \cap T = T$.

(c) (2 points) Using **proof by induction**, prove that the intersection of all sets in a sunflower is the kernel. In other words, the following statement:

$$\forall n \geq 2, \text{a sunflower of size } n \text{ and kernel } T \text{ has } S_1 \cap S_2 \cap \ldots \cap S_n = T$$

Hint: Use part (b) at some point in your inductive step.

Problem 4: They love me, they love me not...

(a) (2 points) We are given a flower with 5 petals, numbered 1, 2, 3, 4, and 5. The flower is 3-plucked if petals, i, $i+1$, and $i+2$ are plucked for some $i \in \{1,2,3\}$. In other words, the flower is 3 plucked if petals 1, 2, and 3 are plucked, **or** if petals 2, 3, and 4 are plucked, **or** if petals 3, 4, and 5 are plucked. How many flowers are 3-plucked?

Hint: Count them using Inclusion-Exclusion. So let S_i, for $i \in \{1,2,3\}$, be the set of all flowers that have petals i, $i+1$, and $i+2$ plucked. For instance, S_1 is the set of all flowers that have petals 1, 2, and 3 plucked (how big is S_1?). Find $|S_1 \cup S_2 \cup S_3|$.

Note: You must solve this problem using Inclusion-Exclusion and show your work.

(b) (2 points) If the flower has 101 petals (yes, that's a flower from another planet), and we pluck any 52 petals that we choose, prove that the flower will have two plucked petals that are next to each other. (In fact, 51 plucked petals would be enough, but harder to prove.)

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Solution

Problem 1

(a) All we need is find some example $n > 1$ such that $F_n = T_n$.

n	0	1	2	3	4	5	6	7	8	9	10	...
F_n	0	1	1	2	3	5	8	13	21	34	55	...

We see that $F_{10} = 1 + 2 + \ldots + 10 = 55 = T_{10}$.

Grading policy: 1 point for finding it. Partial credit of 0.5 points for providing 1 ($n = 2$), 3 ($n = 4$), 21 ($n = 8$) as triangular numbers.

(b) Suppose that there exist x and y such that $x + y$ is the average of all other integers in the set (0.5 point).

Then:

$$x + y = \frac{T_n - (x + y)}{n - 2} \text{ (0.5 point)}$$

$$(x + y)(n - 2) = T_n - (x + y) \text{ (1 point for solving and getting the contradiction)}$$

$$(x + y)(n - 1) = T_n$$

$x+y = T_n/(n-1) = (1/2)n(n+1)/(n-1) \notin \mathbb{N}$ when $n > 3,$ a contradiction since $x+y$ is an integer.

Note: When $n = 3$; in the set $\{1, 2, 3\}$, $1 + 2 = 3$ is the average of all other integers (just 3).

Grading policy: 0.5 points for starting the proof correctly, and another 0.5 points for reasonably exressing the statement mathematically. The last 1 point is for completion of the proof and reaching the contradiction.

Problem 2

(a) The contrapositive is given by

$$\{x, y\} \text{ is colorful} \Rightarrow x + y + xy \text{ Sis odd}$$

To prove this: The set $\{x, y\}$ is colorful $\Rightarrow x$ is even and y is odd (or the other way) $\Rightarrow x = 2k$ and $y = 2k' + 1$ (or the other way) $\Rightarrow x + y + xy = 2k + 2k' + 1 + 2k(2k' + 1) = 2[k + k' + k(2k' + 1)] + 1 = 2k'' + 1 \Rightarrow x + y + xy$ is odd.

Grading policy: 1 point for stating the contrapositive correctly, and 1 point for the proof. For the proof, half a point is given as partial credit is the general idea of dealing with even and odd is there, but the proof is not 100% correct. A proof that argues that xy is even and that even + odd + even is odd, is also acceptable.

(b) Consider $x = 1$ and $y = 3$. Then $\{x, y\}$ is not colorful and $x + y + xy = 7$ is odd. This means the statement

$$\underbrace{\{x, y\} \text{ is not colorful}}_{True} \Rightarrow \underbrace{x + y + xy \text{ is even}}_{False}$$

does not hold, and we can't reverse the implication.

Grading policy: 1 point if everything is correct. Partial credit is hard to assess here, but is left to the grader.

(c) The third argument is the correct. It is true that there is a bijection between $\mathcal{P}(\mathbb{N})$ and E. It is true that if E is uncountable, so must be O (simply replace every even integer n by $n - 1$). Finally, C can be thought of as $E \times O$ by listing the even elements of a set in C first, followed by the odd ones.

One could also say that C is "bigger" than E, because if we drop all odd elements from a set in C, we get a set in E. So C is uncountable.

The first argument is wrong. If a set is a subset of another, it does not mean there is no bijection between the two. Also, the continuum hypothesis does not necessarily exclude the existance of an infinity between $\mathbb{N}$ and its power set. The second argument is not rigorous enough. If C is countable, one may think the same about E and O, but it's not a solid argument. The fourth argument is simply wrong and makes no sense.

Grading policy: 1 point for the third argument with some justification of why it was chosen (0.75 points for simply choosing (3)). Partial credit: 0.25 points for (1), 0.5 points for (2), and 0 points for (4).

Problem 3

(a) The Venn diagram of the sets suggests a sunflower shape where all sets intersect in the same kernel.

Grading policy: Any opinion gets 1 point.

(b) Observe that $T = S_i \cap S_j$ for some $j \neq i$ (by definition of sunflower).

$$S_i \cap T = S_i \cap (S_i \cap S_j) = (S_i \cap S_i) \cap S_j = S_i \cap S_j = T$$

Another possibility: T is a subset of S_i by definition of sunflower, or because $T = S_i \cap S_j$. Therefore, $S_i \cap T = T$.

Grading policy: 1 point for correct proof/argument. Partial credit is hard to assess here, but is left to the grader. For instance, 0.2 for some attempt that is mostly wrong, 0.5 for a vague proof or restating of the question, 0.8 if some detail is incorrect.

(c)

Base case: When $n = 2$, a sunflower of size 2 and kernel T has $S_1 \cap S_2 = T$ by definition of sunflower.

Inductive step:

$$P(k) : \text{A sunflower with size } k \text{ and kernel } T \text{ has } S_1 \cap S_2 \cap \ldots \cap S_k = T \text{ (inductive hypothesis)}$$

$$P(k+1) : \text{A sunflower with size } k+1 \text{ and kernel } T \text{ has } S_1 \cap S_2 \cap \ldots \cap S_{k+1} = T$$

Assuming $P(k)$ is true, we want to show that $P(k+1)$ is also true. Effectively, we show that $P(k) \Rightarrow P(k+1)$.

Given a sunflower $S_1, S_2, \ldots, S_{k+1}$ with kernel T, it must be that $S_1, S_2, \ldots, S_k$ is also a sunflower with kernel T. By the inductive hypothesis, $S_1 \cap S_2 \cap \ldots \cap S_k = T$. Therefore,

$$S_1 \cap S_2 \cap \ldots \cap S_{k+1} = S_1 \cap S_2 \cap \ldots \cap S_k \cap S_{k+1} = (S_1 \cap S_2 \cap \ldots \cap S_k) \cap S_{k+1} = T \cap S_{k+1} \text{ (by inductive hypothesis)}$$

$$= T \text{ (by part (a) above)}$$

Grading policy: 1 point for base case, and 1 point for inductive step. A crucial aspect of the inductive step is the use of $S_1 \cap S_2 \cap \ldots \cap S_k$ as an intermediate intersection to apply the inductive hypothesis. A proof that goes like:

$$S_1 \cap S_2 \cap \ldots \cap S_n = (S_1 \cap S_2) \cap S_3 \cap \ldots \cap S_n = T \cap S_3 \ldots \cap S_n = (T \cap S_3) \cap S_4 \cap \ldots \cap S_n = T \cap S_4 \cap \ldots \cap S_n$$

and unfolds the sum by propagating T is acceptable. Not really proof by induction, so it will get 1.5 points. Some partial credit is given; for instance, if the inductive hyp. is stated 0.5 points, 0.25 points if the inductive hyp. is mentioned, but not correctly stated. In addition, if the base case is incorrect, but everything else was fine, including inductive hyp. and inductive step, only 0.5 points were taken away. If base case is only stated as $n = 2$, without explanation, this was worth only 0.25 points, and 0.5 points were taken away for an incomplete base case explanation (or one that is not entirely correct).

Problem 4

(a) I will use P for plucked. Here are the patterns dictating the size of sets.

$$|S_1| = 4 \quad (PPP**)$$

$$|S_2| = 4 \quad (*PPP*)$$

$$|S_3| = 4 \quad (**PPP)$$

$$|S_1 \cap S_2| = 2 \ (PPPP*)$$

$$|S_1 \cap S_3| = 1 \ (PPPPP)$$

$$|S_2 \cap S_3| = 2 \ (*PPPP)$$

$$|S_1 \cap S_2 \cap S_3| = 1 \ (PPPPP)$$

$$|S_1 \cup S_2 \cup S_3| = |S_1| + |S_2| + |S_3| - |S_1 \cap S_2| - |S_1 \cap S_3| - |S_2 \cap S_3| + |S_1 \cap S_2 \cap S_3|$$

$$= 4 + 4 + 4 - 2 - 1 - 2 + 1 = 8$$

Grading policy: 0.25 points for every correct size of the 7 sizes, and 0.25 points for a correct use of the formula.

(b) We can use pigeonhole. Divide the petals into "boxes" as follows:

$$\underbrace{(1,2)}\ \underbrace{(3,4)}\ \underbrace{(5,6)}\ \ldots\ \underbrace{(99,100)}\ \underbrace{(101)}$$

These are 51 boxes. If 52 petals (the objects) are chosen to be plucked, two of them must belong to the same box by the pigeonhole principle. Therefore, two plucked petals will be next to each other.

Grading policy: 1 point for identifying that this is a candidate for pigeonhole, and 1 point for a complete correct proof with 51 boxes and 52 objects. Partial credit was given; for instance, 0.5 points for some explanation without identifying number of boxes and number of objects, and 0.5 points for a correct answer such as $\lceil 52/51 \rceil$ but no explanation.

Note: One can prove that 51 petals are enough to establish the result, using the fact that petals 1 and 101 are next to each other. The proof is similar. First, since only 51 petals are plucked, some petal must remain. Remove that petal from consideration (hide it, cover it with your finger). Now we are left with 100 petals and still 51 of these are plucked. Dividing the 100 petals into 50 boxes (where a box could include petal 101 and petal 1; for instance, if an even petal is the hidden one), each with two petals next to each other, will establish the result. Here's another way: Each space between petals is a box, so we have 101 boxes. If you pluck a petal, place one token in the box to its left, and one token in the box to its right (this changes nothing, but it's a structure on top of the problem). As such, $51 \cdot 2$ tokens are placed. So we place 102 tokens in 101 boxes, and one box will have at least two tokens, which could only mean that two adjacent petals have been plucked.