Discrete Math Test 1 Spring 2025

Saad Mneimneh

March 12, 2025

DO NOT WRITE HERE

Name: Solution

P1:

EmplID:

P2:

Recitation Instructor (circle one):

P3:

Shayan (shokri) on Mon   Anthony on Wed/Thu   Taha on Thu

P4:

Note: Write clearly in the dedicated space,
and use ink to answer the questions, not a pencil.

P5:

Do not turn this page before it's time to start, but while you wait, make sure you wrote your name, and you may draw something in the space below.

Problem 1: Quick and short answers (do not show work)

(a) (1 point) In how many ways can we stack n cubes with different colors? $n!$

(b) (1 point) In how many ways can we pick k out of n movies and rank them? $n!/(n-k)! = \binom{n}{k}k!$

(c) (1 point) In how many ways can we make a pair given n socks? $\binom{n}{2}$

(d) (1 point) How many n-bit binary words exist? 2^n

(e) (1 point) Same as above if in addition we must have exactly k 1s? $\binom{n}{k}$

(f) (1 point) How many k letter words can we make if the alphabet size is n? n^k

(g) (1 point) Same as above if in addition the letters of the word must be in alphabetical order? $\binom{n-1+k}{n-1} = \binom{n-1+k}{k}$

Grading policy: 1 point for each correct answer. Some partial credit was given; for example, if answer is k^n while the correct answer is n^k, or if there is a slight error in some formula.

Answer next to each question, do not show your work.

Problem 2: Musical Chairs (modified from T.H.)
In a musical chair game, there is music, and there are n people and $n-1$ different chairs. When the music stops, each person will try to grab a chair. Therefore, under normal circumstances, one person will remain standing. That's the first round.

(a) (2 points) Let's focus on the first round. Each person grabbed a chair, and two of them ended up sitting on the same one! Someone thought:

In how many ways could this happen?

While we wait to resolve the issue of who gets to sit among the two, answer the above question using the following procedure:

```
                                                                     #ways
   1. choose a person ................................................   n
   2. choose another person ..........................................  n-1
   3. choose a chair and put the two people on it ....................  n-1
   4. put the remaining n-2 people on the remaining n-2 chairs .......  (n-2)!
```

Fill the above, **explain** whether there is overcounting or not (and **adjust** if there is), and **name** the principle behind obtaining the final answer.

Solution: Permuting the choices of the first two phases results in the same two people sitting on the chair. So we overcount by 2. By the product rule, the answer (after adjusting for overcounting) is $n(n-1)(n-1)(n-2)!$. This is equivalent to $n!(n-1)/2$ and $\binom{n}{2}(n-1)!$.

Grading policy: 0.25 points were given to each correct entry in the table above. Another 0.5 points for correct overcounting adjustment. Another 0.5 points for mentioning the product rule as the principle used (0.25 points only if not explicitly mentioned but product was used).

(b) (2 points) The actual game consists of multiple rounds as follows: At the end of each round, the person who remain standing leaves the game and takes away a chair (so we say the two are paired). The music resumes, and this process is repeated until one person (the winner) and zero chairs remain in the game. In how many ways can people be paired to chairs when this musical chairs game ends?

Explain your thought process and provide insight about **why** the answer is not $(n-1)!$.

Grading policy: 0.5 points were given for explaining why correct the answer is not $(n-1)!$. The remaining 1.5 points were divided generally as follows: 0.5 points for any thought process that makes some sense. 0.5 points for the correct answer. 0.5 points for mentioning that we are seating n people on $n-1$ chairs.

Solution: The whole game is nothing but a complicated process for seating n people on $k = n-1$ chairs. That's a k-permutation. It can be done in $n!/[n-(n-1)]! = n!/1! = n!$ ways. Another way to think about this is as follows:

```
                                                         #ways
   1. choose a person to be out of the game ........      n
   2. put the reamining n-1 people on n-1 chairs ...   (n-1)!
```

By the product rule, this is $n!$. The answer is not $(n-1)!$ because we are not simply seating $(n-1)$ people on $(n-1)$ chairs. We are also leaving one person out.

Problem 3: Pokémon Go (modified from Z.T.P)
A Pokémon card pack contains 10 cards, and there may be multiples of the same cards in the pack. For example, we can have a pack with 1 Pikachu, 2 Bulbasaur, 3 Charmander, and 4 Squirtle cards ($1+2+3+4 = 10$). Note that this pack is different from another pack that contains 1 Charmander, 2 Squirtle, 3 Pikachu, and 4 Bulbasaur cards.

(a) (2 points) If there are 151 unique Pokémon cards, how many different packs can we make? In addition, **describe** which of the 4 selection scenarios this corresponds to?

Solution: This is a selection with repetition and no order. We are selecting 10 from 151, so it can be done in $\binom{151-1+10}{151-1} = \binom{160}{150} = \binom{160}{10}$ ways. Another interpretation is given by integers solutions to:

$$x_1 + x_2 + \ldots + x_{151} = 10$$

where $x_i \geq 0$ for all i, and x_i represents the multiplicity of Pokémon i.

Grading policy: 0.5 points were given for identifying there is repetition, and 0.5 points for identifying order is not relevant. The remaining one 1 is given for a correct answer, with possibility of partial credit is expression is slightly wrong.

(b) (2 points) We now have a new rule. The pack may contain at most 1 Picachu, but if it does contain a Picachu, then all other cards must be distinct. How many packs can we make?

Solve this version of the problem in **two cases** based on the existence of a Picachu, and **explain** which principle is needed to combine the results and why.

Solution:
case 1: Picachu exists: then choose 9 different cards from 150 (no repetition and no order): $\binom{150}{9}$.
case 2: No Picachu: then choose 10 cards from 150 with repetition, and still no order: $\binom{150-1+10}{150-1} = \binom{159}{149}$.

Since the two categories are disjoint, by the addition rule we have

$$\binom{150}{9} + \binom{159}{10}$$

Grading policy: 0.5 points were given for the explicit mention of the addition rule. The remaining 1.5 points were reserved for the answer in each case and producing the final result.

Problem 4: The many faces of Discrete Math (modified from N.S.)

Consider the *word* "DISCRETEMATH".

(a) (2 points) How many anagrams of that word can we make? No need to show work here, but simply write down the answer and **briefly explain** why it is not 12!

Solution: We have $\frac{12!}{2!2!} = 12!/4$ anagrams. The answer is not 12! because 12! corresponds to all permutations, and since some letters repeat, some permutations produce the same anagram.

Grading policy: The correct answer with explanation received 2 points. Points can be taken off depending on the explanation or if it's missing. The answer $14!/(2 \cdot 2)$ received partial credit due to a typo that was corrected in class.

(b) (2 points) How many anagrams can we make if we allow up to three words. For instance, we can insert zero spaces, one space, or two spaces. For example, "DISCRETE MATH" is allowed by inserting one space, and "DISC MATH TREE" is also allowed by inserting two spaces. We can't insert spaces at the beginning or end, and we can't have two consecutive spaces.

Hint: To count anagrams with multiple words, first construct the one word anagram, then consider the number of ways you can insert spaces within that one word.

Show your work.

<u>case 1</u>: No spaces: $\frac{12!}{4}$ (simply choose an anagram)

<u>case 2</u>: One space: $\frac{12!}{4}\binom{11}{1}$ (choose an anagram, then choose 1 space from 11 spaces)

<u>case 2</u>: Two spaces: $\frac{12!}{4}\binom{11}{2}$ (choose an anagram, then choose 2 spaces from 11 spaces)

By the addition rule: $\frac{12!}{4}\left[\binom{11}{0} + \binom{11}{1} + \binom{11}{2}\right]$.

Grading policy: Each correct case received 0.5 points. Putting the whole thing together using addition gets another 0.5 points.

Problem 5: Subsets and Binomial coefficients (modified from V.V.)
Consider the set $S = \{1, 2, 3, \ldots, n\}$ and let k be a fixed integer.

(a) (1 point) For any given i, consider the set T_i of all subsets of $\{1, 2, \ldots, i-1\}$ of size $k-1$. How big is T_i? (Simply write down the answer; this is not a trick question and has nothing to do with the set S yet.)

Solution: There are $\binom{i-1}{k-1}$ subsets of $\{1, 2, \ldots, i-1\}$ of size $k-1$.

Grading policy: 1 point for correct answer.

(b) (1 point) For any given i, let M_i be the set of all subsets of S of size k that have i as the largest element. Consider function f below:

$$f : T_i \to M_i$$

$$f(x) = x \cup \{i\}$$

Illustrate f for $k = 3$ and $i = 5$ by showing the domain, the co-domain, and the arrows mapping elements from the domain to the co-domain.

Solution: When $k = 3$ and $i = 5$, T_5 is the set of all subsets of $\{1, 2, 3, 4\}$ of size 2, and M_5 is the set of all subsets of S of size 3, where 5 is the largest.

```
{1,2}  --->  {1,2,5}
{1,3}  --->  {1,3,5}
{1,4}  --->  {1,4,5}
{2,3}  --->  {2,3,5}
{2,4}  --->  {2,4,5}
{3,4}  --->  {3,4,5}
         f
 T_5          M_5
```

Grading policy: 1 point for providing reasonable illustration, minor mistakes forgiven.

(c) (2 points) Show that $|T_i| = |M_i|$ by considering properties of f. What do you need to show about function f? (**Explain and show it**).

Solution: To show that the two sets have the same size, we need to show that f is a bijection. Therefore, we need to show that f is one-to-one and onto:

<u>one-to-one</u>: $f(x_1) = f(x_2) \Rightarrow x_1 \cup \{i\} = x_2 \cup \{i\} \Rightarrow x_1 = x_2$, since both x_1 and x_2 do not contain i.

<u>onto</u>: Given $y \in M_i$, let $x = y - \{i\}$. By construction, $f(x) = x \cup \{i\} = y$, and $x \in T_i$ because all elements of x are less than i (since i was the largest in y).

Grading policy: 1 point for showing one-to-one, and 1 point for showing onto. Possible partial credit.

(d) (2 points) By summing up your answer of part (a) over all i, discover an identity involving the binomial coefficients. Express your answer as a sum notation. It should look like:

$$\sum_{\ldots}^{\ldots} \ldots = \ldots$$

Briefly explain how you obtained the right-hand side. *Hint*: Observe that the sets $M_1, M_2, \ldots, M_n$ are all disjoint, so: $\sum_{i=1}^{n} |M_i| = |M_1 \cup M_2 \cup \ldots \cup M_n|$ (and what does that union represent?).

Solution:

$$\sum_{i=1}^{n} \binom{i-1}{k-1} = \sum_{i=1}^{n} |T_i| = \sum_{i=1}^{n} |M_i| = |M_1 \cup M_2 \cup \ldots \cup M_n| = \binom{n}{k}$$

The first equality is given by part (a), the second by part (c), the third by the hint, and the last by observing that the union $M_1 \cup M_2 \cup \ldots \cup M_n$ represents all subsets of S of size k, since every subset of size k must have some largest element. Therefore,

$$\sum_{i=1}^{n} \binom{i-1}{k-1} = \binom{n}{k}$$

Remark: Since $\binom{n}{k} = 0$ when $k > n$, the above sum can start at $i = k$. We get

$$\sum_{i=k}^{n} \binom{i-1}{k-1} = \binom{n}{k}$$

$$\binom{k-1}{k-1} + \binom{k}{k-1} + \ldots + \binom{n-1}{k-1} = \binom{n}{k}$$

Grading policy: 0.5 points for explaining, but a correct answer received 1 point. Some partial credit when possible.