CSCI 150 Discrete Mathematics
Final Test

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There will be no D as a final letter grade. A D grade will automatically turn into an F. The possible grades are A+, A, A-, B+, B, B-, C+, C, and F.
If you don't want your D to become F, check this box ☐.

Name: Solution EmplID:

Recitation instructor (circle one): **Shayan** **Arezoo** **Daniel** **Anthony**
Recitation section (circle one):

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Shayan:   Mon 11:30   Mon 12:30   Mon 3:00   Mon 4:00
Arezoo:   Wed 9:30    Wed 10:30   Wed 11:30  Thu 4:00
Daniel:   Thu 11:30   Thu 12:30
Anthony:  Tue 2:30    Thu 3:00
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Dont' write in this section

    P1:

    P2:

    P3:

    P4:
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 Total:
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Write your name on this page.

Don't turn the page until it's time.

There are 10 pages (including this one).

There are 4 problems (with multiple parts each).

Turn all your cell phones off and place them away (no calculators or smart watches either).

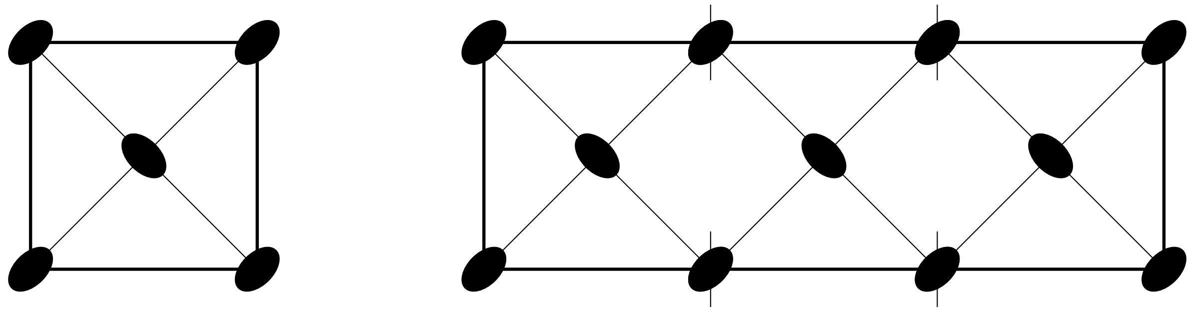
If you need to leave (e.g. bathroom break), please give me your test and all your cell phones.

There is a total of 35 points, but everyone will get the point for Problem 1(d) regardless of how they answer. Therefore, the test is designed so that the lowest possible grade is 1/35 and the highest possible grade is 35/35.

Write your answers neatly and clearly. Do no squeeze your answers between questions, use the dedicated space for each problem. Make sure everything is legible.

FYI: I tried my best to design questions that (1) cover most of the concepts we have seen, (2) mimic several ideas in recitations, homework, and sample test questions, and (3) present non-trivial but reasonable problems.

Problem 1: Kufiya
A *Kufiya* is a pattern that resembles a fishnet. We can *Kufiyize* any $1 \times n$ rectangle, as shown below for $n = 1$ (left) and $n = 3$ (right), dividing the rectangle into triangular and square areas. For instance, there are 4 areas in total when $n = 1$, and 10 areas when $n = 3$ (count them to confirm).



Let R_n for $n \in \mathbb{N} = \{1, 2, 3, \ldots\}$ be the total number of areas (both triangular and square) in a *kufiyized* rectangle of length n. For example, $R_1 = 4$ and $R_3 = 10$.

(a) (1 point) Express R_n in terms of R_{n-1}. Justify your answer.

Solution: This describes R_n in terms of R_{n-1}: $R_n = R_{n-1} + 3$. The justification is the following: When we increase the length of the rectangle by 1, it's like adding a square with $R_1 = 4$ areas. But one triangular area of the square merges with an existing triangular area of the rectangle (the last area), making a combined square area, and leading to a net of three new areas.

(b) (3 points) Transform your recurrence from part (a) into a homogeneous recurrence that expresses R_n in terms of R_{n-1} and R_{n-2}. Then use the characteristic equation method to find R_n as a function of n.

Note: Part (b) may not the best approach to obtain R_n, but it uses the concept of a linear homogeneous recurrence. Partial credit will be given if an alternative method is used. For instance, guessing an expression for R_n and proving it by induction.

Solution: We have $R_n = R_{n-1} + 3$ and $R_{n-1} = R_{n-2} + 3$. Subtracting the two equations we get $R_n - R_{n-1} = R_{n-1} - R_{n-2}$, leading to $R_n = 2R_{n-1} - R_{n-2}$. The characteristic equation is $x^2 - 2x + 1 = 0$ or $(x-1)^2 = 0$, leading to two solutions that are equal $p = q = 1$ and, therefore, R_n has the form: $R_n = c_1(p)^n + c_2 n(p)^n = c_1 + c_2 n$. Using R_1 and R_3, we get that $c_1 = 1$ and $c_2 = 3$. Finally, $R_n = 3n + 1$.

Alternative solutions:
Rate of growth: Since $R_n = R_{n-1} + 3$, R_n grows with n at a rate of 3. Therefore, $R_n = 3n + c$. To make $R_1 = 4$, we choose $c = 1$.

Telescoping sum:

$R_1 = 4$
$R_2 - R_1 = 3$
$R_3 - R_2 = 3$
$\vdots$
$R_n - R_{n-1} = 3$

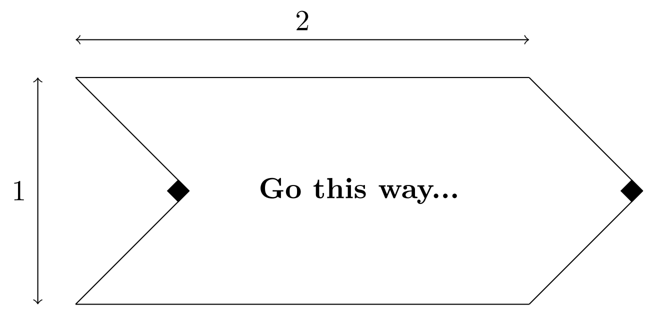
Adding the equations, we get:

$$R_n = 4 + 3(n-1) = 3n + 1$$

Induction: Guess that $R_n = 3n + 1$ by observing few examples (where R_1 could serve as the base case). Given that $R_k = 3k + 1$ as the inductive hypothesis, the inductive step will be:

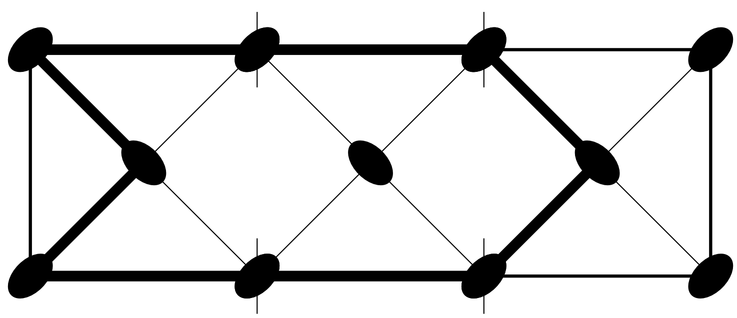
$$R_{k+1} = R_k + 3 = (3k+1) + 3 = 3(k+1) + 1$$

(c) (2 points) As you might know, many university campuses are witnessing student protests. A sign to guide students where to go has been designed, with dimensions in feet as show below. A car passes by and splashes the sign with 7 spots of mud. Show that two of the spots must be within 1 foot of each other (you may think of the mud spots as points).



Hint: Think about a *Kufiya* pattern inside the sign.

Solution: There are 6 areas within the sign after Kufiyizing it.



By the pigeonhole principle, $\lceil 7/6 \rceil = 2$ spots will be in the same area. Each area is either a triangle or a square. The largest distance in the triangle is the largest side, which is 1. Similarly, the largest distance in the square is the diagonal, which is also 1. This gives the result.

(d) (1 point) How have you been affected by the recent events in the Middle East and their consequences as experienced in the United States? You can answer in any way you want, including a blank answer. You may also provide an opinion if you wish. You will get the 1 point regardless of how you answer. Think of this as an opportunity to say anything you like. It will not be used.

[Use this page and next (front and back) to answer questions of Problem 1]

Problem 2: Proofs
In this problem, you are asked to provide proofs. Please by clear and neat because it is hard to grade proofs. It is very likely that proofs will be graded with no partial credit. A clear/clean/correct proof will simply present itself immediately to the reader.

(a) (2 points) Let n be an integer. Prove that $n(n+1)/2$ is always an integer.

Solution: We can consider two cases based on whether n is even or odd:

- $n = 2k \Rightarrow n(n+1)/2 = 2k(2k+1)/2 = k(2k+1) \in \mathbb{Z}$.
- $n = 2k+1 \Rightarrow n(n+1)/2 = (2k+1)(2k+2)/2 = 2(2k+1)(k+1)/2 = (2k+1)(k+1) \in \mathbb{Z}$

Another way to say this informally is like this: either n or $n+1$ is even. So either $n/2$ or $(n+1)/2$ is an integer.

(b) (2 points) Let $T_0 = 0$ and define $T_n = 0T^*_{n-1}0$, where T^*_n is obtained by flipping the bits of T_n (i.e changing a 0 into a 1 and vice-versa).

Prove by induction that for all $n \in \mathbb{N} \cup \{0\}$, T_n has n 1s and $n+1$ 0s. Therefore,

$$P(n) : T_n \text{ has } n \text{ 1s and } n+1 \text{ 0s}$$

Solution:
Base case: T_0 has 0 1s and 1 0. So $P(0)$ is true.

Inductive Step: Assume $P(k)$ is true, and let's consider $P(k+1)$. We know $T_{k+1} = 0T^*_k0$. Since T_k has k 1s and $k+1$ 0s by the inductive hypothesis, T^*_k has $k+1$ 1s and k 0s. Therefore, $T_{k+1} = 0T^*_k0$ has $k+1$ 1s and $k+2$ 0s. Done.

(c) (2 points) I imagined Pythagoras saying: "There are no isosceles right triangles with integer sides". Let a, b, and c be integers. Prove by contradiction the following statement:

$$a^2 + b^2 = c^2 \Rightarrow a \neq b$$

Hint: Recall that the negation of $P \Rightarrow Q$ is $P \wedge \neg Q$.

Solution: Assume $a^2 + b^2 = c^2$ and $a = b$.

$$a^2 + b^2 = c^2 \wedge a = b \Rightarrow a^2 + a^2 = c^2 \Rightarrow 2a^2 = c^2 \Rightarrow 2 = c^2/a^2 \Rightarrow \sqrt{2} = c/a$$

We have a contradiction, since $\sqrt{2}$ is irrational.

(d) (2 points) Given integers a and b, prove using the contrapositive that:

$$a + b \geq 15 \Rightarrow a \geq 6 \;\vee\; b \geq 10$$

Hint: Use DeMorgan's law to negate.

Solution:

$$\underline{a < 6 \wedge b < 10} \Rightarrow a \leq 5 \wedge b \leq 9 \Rightarrow a + b \leq 14 \Rightarrow \underline{a + b < 15}$$

(e) (2 points) Prove that there exists a Fibonacci number greater than 1 such that

$$F_n = \sum_{i=1}^{n} i$$

Note: Recall that the Fibonacci sequence is given by $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$.

Solution:

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$$

$$F_{10} = 55 = \sum_{i=1}^{10} i = 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10$$

[Use this page and next (front and back) to answer questions of Problem 2]

Problem 3: A tree with flowers

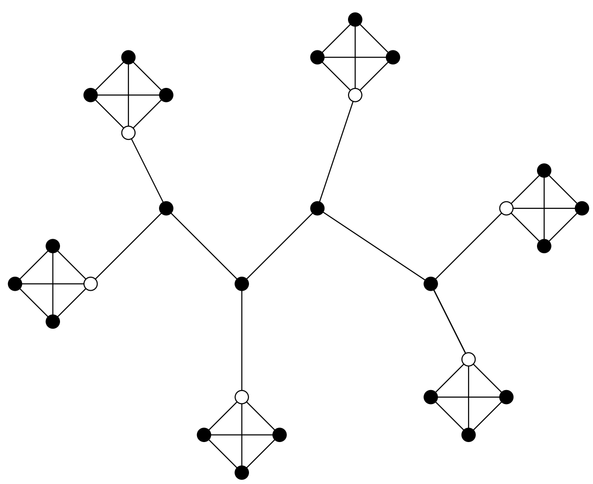
Consider a tree that satisfies the following condition: every vertex has degree either 3 or 1. A vertex with degree 1 is called a *leaf*.

(a) (3 points) How many edges are there in total if there are n leaves (the answer must be in terms of n)?

Hint: Let m be the number of non-leaves. Use the Handshake Lemma and a property of trees. For instance, you can find m in terms of n, then find the total number of edges.

Solution:
Handshake Lemma: $n + 3m = 2e = 2(n + m - 1)$, where the last equality follows from the fact that in a tree the number of edges is the number of vertices minus 1. This gives $m = n - 2$. Therefore, the total number of edges is $n + n - 2 - 1 = 2n - 3$.

Given a tree as described above, we create a tree with flowers. A flower is the complete graph on 4 vertices, known as K_4. As a result of this *blooming*, each leaf is replaced by K_4. An example of a tree with 6 flowers is shown below (the original tree leaves are shown in white):



(b) (2 points) Consider a tree with n flowers. Is it planar? Whether your answer is YES or NO, provide a justification. In addition, if your answer is YES, find how many faces a tree with n flowers should have (the answer must be in terms of n).

Solution: A tree with flowers is planar. Each flower can be drawn without crossing by rerouting one of the edges "inside" the flower to the outside. Each flower will have 3 faces, in addition to the common outer face which all flowers share. The number of faces is, therefore, $3n + 1$.

Another argument for planarity is that the tree with flowers does not contain the "shapes" $K_{3,3}$ or K_5.

(c) (3 points) Let V be the set of vertices in a tree with n flowers. We say that a vertex $u \in V$ is in a flower iff it belongs to some K_4. Consider a function $f : V \to V$. How many such functions are bijections? (You can express your answer in terms of $|V|$.) **(1 point)**.

Solution: The number of bijections from V to V is $|V|!$, since each bijection can be thought of as a permutation.

If we assume that f is such that:

$$u \text{ is in a flower} \Rightarrow f(u) \text{ is in the same flower}$$

How many bijections $f : V \to V$ are there that satisfy the above? (You can express your answer in terms of n and/or $|V|$, whatever you find convenient.) **(2 points)**

Solution: Given such a bijection, there are $|V| - 4n$ vertices not in flowers, which can be permuted in $(|V| - 4n)!$ ways. For each flower, the 4 vertices can be permuted in $4! = 24$ ways, leading to 24^n possible ways of mapping vertices inside flowers. By the product rule, this is

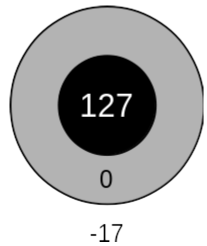
$$24^n(|V| - 4n)!$$

Also one can observe that $|V| - 4n = m = n - 2$, so we have $24^n(n - 2)!$.

[Use this page and next (front and back) to answer questions of Problem 1]

Problem 4

Consider the following target. We have 143 **identical** darts to throw at the target. If a dart lands on black, it contributes 127 points. If a dart lands on gray, it contributes 0 points. If a dart lands on white (misses the target), it contributes -17 points. The score is the total points of all 143 darts.



Observation (that you don't have to prove): Because 127 and 17 are co-prime, every achievable score is achieved in exactly one way of placing the identical 143 darts on the three colors of the target.

(a) (3 points) How many possible scores are there? *Hint*: Based on the above observation, all we need is to find the number of possible placements of 143 identical darts on three colors.

Solution: This is like selecting 143 out of 3 with repetition and no order. Or, the number of solutions to $x_1 + x_2 + x_3 = 143$, where $n = 3$ and $k = 143$. The answer is $\binom{n+k-1}{n-1} = \binom{145}{2} = \frac{145 \cdot 144}{2} = 10,440$.

(b) (2 points) Each dart gives the color black, gray, or white, depending on where it lands. Therefore, the 143 darts also define an *ordered* sequence of colors: color of dart 1, color of dart 2, ..., color of dart 143. Let a good color sequence be a sequence where all three colors show. How many good sequences are there?

Hint: First count bad sequences using inclusion-exclusion, where you define S_i as the set of sequences that miss color i. Then subtract the answer from the total number of sequences. Express your answer in the following format:

$$\square - (\square + \square + \square - \square - \square - \square + \square)$$

Solution: There are x^n sequences of length n using x colors. So, we will end up with:

$$3^{143} - (2^{143} + 2^{143} + 2^{143} - 1^{143} - 1^{143} - 1^{143} + 0^{143})$$

$$3^{143} - (2^{143} + 2^{143} + 2^{143} - 1 - 1 - 1 + 0)$$

(c) (2 points) A player was losing precision over times. So the first 82 darts landed on black, the next 41 darts landed on gray, and the final 20 darts landed on white (missed). This defines the color sequence:

$$\underbrace{b \ldots b}_{82}\underbrace{g \ldots g}_{41}\underbrace{w \ldots w}_{20}$$

How many color sequences (including the above one) achieve the player's exact score? (This is not a contradiction, since all these sequences correspond to the same placement of darts on colors.)

Solution: These are like anagrams, we have $\frac{143!}{82!41!20!}$ anagrams.

(d) (3 points) Assume now that you can place the darts exactly where you want (so you don't even have to aim and throw). Describe how you can achieve a total score of 1 (show your work and state how many darts you will place in each region).

Hint: If r, w, and s represent the number of darts that fall in the black, gray, and white region, respectively, then the score is given by $127r + 0w - 17s = 127r - 17s$ (does this ring a bell?).

Solution: The Euclidean algorithm will show us how to make $127r - 17s = 1$.

127	17	8	1	0
1	0	1	-2	
0	1	-7	15	

Therefore, one can write

$$127(-2) + 17(15) = 1$$

$$127(-2 + 17) + 17(15 - 127) = 1$$

$$127(15) - 17(112) = 1$$

We need 15 darts on black, 112 on gray, and $143 - 15 - 112 = 16$ on white.

[Use this page and next (front and back) to answer questions of Problem 4]

SCRATCH (will not be graded)