CSCI 150 Discrete Mathematics
Homework 1 Solution

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Solution

Exercises

1. Express the sum $1+3+5+7+\ldots+127$ using the notation $\sum_{i=a}^{b} f(i)$. First, figure out the number of terms in the sum, then use this information to determine the lower and upper bounds. For instance, a could be 1 and b could be the number of terms, or a could be 0 and be b could be the number of terms minus 1. Finally, find the expression inside the sum that would result in the desired terms. Once you figure out your sum notation, find the answer using the technique of splitting the sum.

   Solution

   Since we step by 2, the number of terms in the sum is given by $(127-1)/2+1=64$. Therefore, we can make our sum notation

   $$\sum_{i=1}^{64} f(i)$$

   or

   $$\sum_{i=0}^{63} g(i)$$

In the first case, $f(i) = 2i - 1$ correctly generates the terms. In the second case, $g(i) = 2i + 1$ works. Let's pick the first case:

$$\sum_{i=1}^{64}(2i-1) = \sum_{i=1}^{64} 2i - \sum_{i=1}^{64} 1 = 2\sum_{i=1}^{64} i - \sum_{i=1}^{64} 1 = 2\frac{64 \cdot 65}{2} - 64 = 64(65-1) = 64^2$$

Note 1: Can you generalize an observation about the sum of the first n odd numbers?

Note 2: Try to get the same answer with $\sum_{i=0}^{63}(2i+1)$.

2. Let k be an integer. When can we claim the following?

$$k^2 = \sum_{i=1}^{k} k$$

Experiment with that expression for several values of k before answering the question.

Note: You might try encoding this as a for loop in Python.

Solution

The above is true only when $k \geq 0$. This is because

$$\sum_{i=1}^{k} k = \underbrace{k + k + \ldots + k}_{k \text{ times}} = k \cdot k = k^2$$

When k is negative, k^2 is positive, but the sum is 0. This is because when the upper bound is smaller than the lower bound in a sum notation, we have an empty sum, which is zero.

3. Assume x and y are fixed parameters. Express the following using a $\prod$ notation:

$$\text{Pock}(n) = (1-x)(1-xy)(1-xy^2)\ldots(1-xy^{n-1})$$

What is Pock(0) and why?

Solution

$$\text{Pock}(n) = \prod_{i=1}^{n}(1-xy^{i-1}) \left(\text{this is also equal to} \prod_{i=0}^{n-1}(1-xy^{i})\right)$$

Observe that Pock(0) = 1 since it represents an empty product (upper bound is $n = 0$, and lower bound is 1).

4. How many five-digit numbers can we make if we can use each of the digits 1, 3, 5, 7, and 9 exactly once? Optional: What if we change the digits to 0, 2, 4, 6, and 8?

   Solution

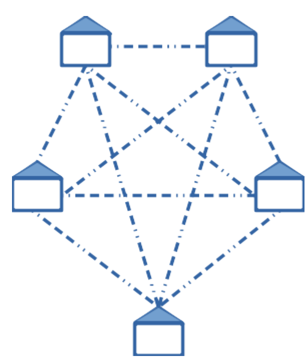
   Each five-digit number is given by a permutation of the digits, so we have 5! numbers. If we change the digits to 0, 2, 4, 6, and 8, we can still apply the same idea; however, every permutation that starts with 0 will not count, since a leading 0 will not make a 5-digit number. Therefore, we can subtract from the number of permutations, which is 5!, all those permutations that start with 0. How many permutations start with 0? Well, we have to permute 2, 4, 6, and 8 in 4! ways. So the answer is $5! - 4! = 4!(5-1) = 4 \cdot 4!$.

   We can interpret the result by using the product rule.

   - choose a digit for the first position: 4 ways (can't choose 0)
   - choose a digit for the second position: 4 ways
   - choose a digit for the third position: 3 ways
   - choose a digit for the fourth position: 2 ways
   - choose a digit for the fifth position: 1 way

   Multiplying reveals $4 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 4 \cdot 4!$.

5. Given the following map, in how many ways can you visit all the houses (each house must be visited exactly once, and you can start with anyone you want)? (why are the answers to this question and the previous question the same?)



Extra challenge (you don't have to submit this): Repeat the above if the road between two of the houses is closed (it does not matter which two).

Solution

This also represents a permutation for $n = 5$, so the answer is also 5!.

Now let's say the road between house 1 and house 2 is closed. This means we have to subtract all permutations that make 1 and 2 adjacent. Well, let's see how many of those we have:

$$12 - - -$$
$$-12 - -$$
$$- - 12-$$
$$- - -12$$
$$21 - - -$$
$$-21 - -$$
$$- - 21-$$
$$- - -21$$

For each of the above pattern, we can permute the other three houses in 3! ways. So we have $8 \cdot 3!$ permutations that make 1 and 2 adjacent. The answer will be $5! - 2 \cdot 4 \cdot 3! = 5! - 2 \cdot 4! = 4!(5-2) = 3 \cdot 4!$.

Another way to see this is through the product rule.

- permute 1, 3, 4, and 5: 4! ways
- choose a position for 2 that is not adjacent to 1 from left or right: 3 ways

Multiplying gives us $3 \cdot 4!$.

Note: Can you generalize this to n houses and the answer $(n-2) \cdot (n-1)!$?

6. The Zelder family has a newborn. The parents want to choose a name and a middle name in such a way that the initials will make three unique letters that appear in alphabetical order (call this a monogram):

$$- - Z$$

For instance ABZ is allowed, but AAZ (not unique) and BAZ (not alphabetical) are not. In how many ways can the parents choose the monogram? (*Hint*: Recall the example of placing one snake on a board where we divided the snakes into disjoint categories based on the head of the snake. Do the same here, and divide the monograms into disjoint categories based on which letter they start with).

Solution

If we start with A, the second letter can be $B, C, \ldots, Y$, and that's 24 possibilities. If we start with B, the second letter can be $C, D, \ldots, Y$, and that's 23 possibilities. We continue this way until the category of starting with X, where the second letter can only be Y, and that's just 1 possibility. By the addition rule, we have $24 + 23 + \ldots + 1 = 24 \cdot 25/2$ possible monograms.

7. Repeat the above if letters of the monogram are not required to be in alphabetical order.

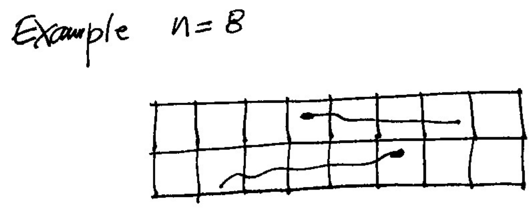
Solution

If we start with A, the second letter can be $B, C, \ldots, Y$, and that's 24 possibilities. If we start with B, the second letter can be $A, C, D, \ldots, Y$, and that's also 24 possibilities. We continue this way until the category of starting with Y, where the second letter can be $A, B, C, \ldots, X$, and that's 24 possibility. By the addition rule, we have $24 + 24 + \ldots + 24$ (25 times). So the answer is $24 \cdot 25$ possible monograms.

Note: You might want to think about two other scenarios: letters are not necessarily unique but appear in alphabetical order, and letters have no constraints.

Problem

Consider a regular $2 \times n$ snakes and ladders board. We are interested in placing two snakes, but each must be in a separate row (the rule that head must be greater than tail is still required). An example is shown below:



(a) Use the product rule to figure out the number of ways can we place the two snakes. Explain your reasoning. The answer must be for a general n.

Hint: There are multiple ways of doing this, but mostly the product rule should be helpful.

Solution

Using the product rule, we can place the two snakes in 4 phases:

(first snake)
- choose a square in the first row: n ways
- choose another square in the first row: $n-1$ ways

(second snake)
- choose a square in the second row: n ways
- choose another square in the second row: $n-1$ ways

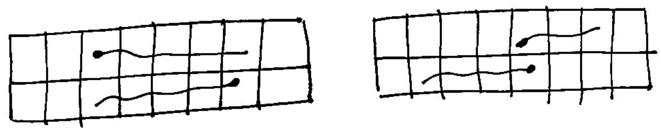
By multiplying, we get $n(n-1)n(n-1)$. Observe that this process overcounts in the usual way. Permuting the first two choices results in the same first snake. Similarly, permuting the last two choices results in the same second snake. So we overcount by $2 \cdot 2 = 4$. The answer is:

$$\frac{n(n-1)n(n-1)}{4} = \frac{n(n-1)}{2}\frac{n(n-1)}{2} = \binom{n}{2}\binom{n}{2} = \binom{n}{2}^2$$

Note: Could you have obtained $\binom{n}{2}\binom{n}{2}$ by thinking a little differently?

(b) The board has n columns. If a head or a tail of any snake falls in a given column, we say that the column is *involved*. In the example above, four columns are involved. Give an example of placing the two snakes where two columns are involved, and another one where three columns are involved.

Solution



(c) Find the number of ways you can place the two snakes if

- two columns are involved
- three columns are involved
- four columns are involved

Solution

Two columns involved: We simply choose two columns out of the n. This completely defines the snakes.

- choose a column: n ways
- choose another column: $n-1$ ways

By the product rule, we have $n(n-1)$ with the usual overcounting by 2. The answer is $n(n-1)/2 = \binom{n}{2}$.

Three columns involved: We choose a common column, and an additional square for each snake.

- choose a column: n ways
- choose a square in the first row (not part of column): $n-1$ ways
- choose a square in the second row (not part of both columns): $n-2$ ways

By the product rule, we have $n(n-1)(n-2)$, and there is no overcounting since an outcome can only be generated in one way (the phases cannot be permuted).

Four columns involved: We choose four squares (two in each row) that are not part of the same columns.

- choose a square in 1st row: n ways
- choose another square in 1st row: $n-1$ ways
- choose a square in 2nd row (not part of previous columns): $n-2$ ways
- choose another square in 2nd row (not part of previous columns): $n-3$ ways

By the product rule, we have $n(n-1)(n-2)(n-3)$. But we have the usual overcounting. Permuting the first two choices results in the same first snake and, similarly, permuting the last two choices results in the same second snake. So we overcount by $2 \cdot 2 = 4$. The answer is $n(n-1)(n-2)(n-3)/4$.

(d) What does the addition rule guarantee about your answer for (a) and the three answers in (c)? Verify.

Solution
Since the three above categories are disjoint, by the addition rule:

$$\binom{n}{2}^2 = \frac{n(n-1)}{2} + n(n-1)(n-2) + \frac{n(n-1)(n-2)(n-3)}{4}$$

which can be easily verified.

As we will see later, $\binom{n}{3} = n(n-1)(n-2)/6$ and $\binom{n}{4} = n(n-1)(n-2)(n-3)/24$. Therefore, we can write the above as:

$$\binom{n}{2}^2 = \binom{n}{2} + 6\binom{n}{3} + 6\binom{n}{4}$$