© Copyright 2024 Saad Mneimneh
It's illegal to upload this document or a picture of it on any third party website

CSCI 150 Discrete Mathematics
Homework 2

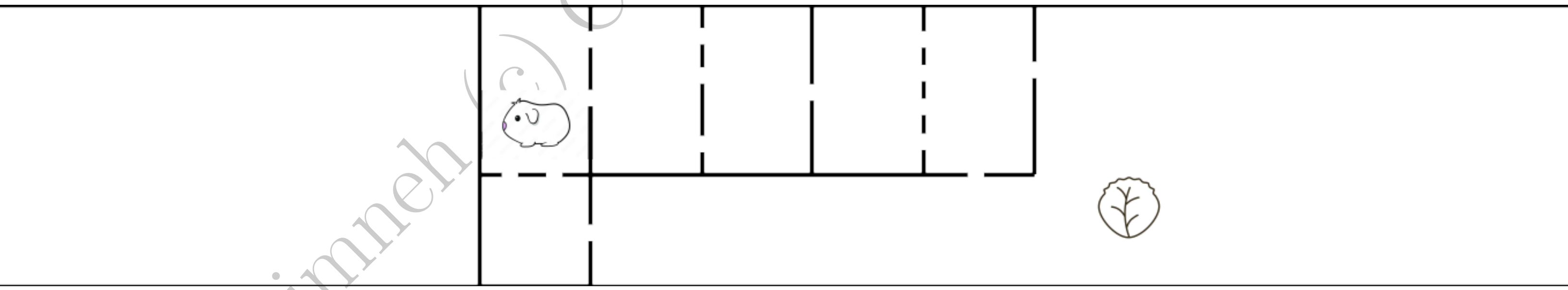
Saad Mneimneh, Computer Science, Hunter College of CUNY

Solution

Saad Mneimneh © Copyright Material - Illegal To Post

Exercises

1. Find in how many ways my guinea pig Chanel can get to the lettuce. Which two important principles have you used to find the answer? (if you have not used some important principles, then rethink the problem)



Solution

The guinea pig can exit either through the upper enclosure, or through the lower enclosure. Let S be the set of all paths that exit through the upper enclosure, and T be the set of all paths that exit through the lower. The sets S and T are disjoint, so we examine $|S| + |T|$ by the *addition rule.*

Now, by the *product rule* and multiplying the number of ways the bunny can exit each successive enclosure, $|S| = 2 \cdot 3 \cdot 1 \cdot 4 \cdot 2 = 48$. Similarly $|T| = 2 \cdot 1 = 2$.

So the total number of ways becomes $48 + 2 = 50$.

2. How many patterns can you make with 2 digits, followed by a letter, followed by a digit, if your pattern cannot start with 0?

   Solution

   By the *product rule*, we have $9 \cdot 10 \cdot 26 \cdot 10 = 23,400$.

3. We place 561 points on the circumference of a circle. How many chords of the circle can these points make?

   Solution

   This is essentially asking for the number of unordered pairs we can make given 561 points, because a chord of the circle is simply a segment joining two points on the circle.

   $\binom{561}{2} = \dfrac{561 \cdot 560}{2} = 157,080$

4. Consider a lottery ticket with 60 numbers. To buy a ticket, you need to choose 5 of these numbers. Each lottery ticket costs \$1. How big must the prize be to justify buying all possible tickets?

   Solution

   The assumption here is that each number can be chosen at most once, and the order of the numbers does not matter.

   $\binom{60}{5} = \dfrac{60 \cdot 59 \cdot 58 \cdot 57 \cdot 56}{5!} = 5,461,512$

5. In the lottery above, how many possible tickets are there if you must choose a sixth number for the "power ball"? Find the answer using the product rule in two ways:

   - choose the power ball first, then your five numbers
   - choose your five numbers first, then the power ball

   Verify that both ways yield the same answer and state an equality that generalizes the result by changing 60 to n and 5 to k. Verify this equality algebraically.

Saad Mneimneh (c) Copyright Material - Illegal To Post

Solution

Approach 1: # ways

1. Choose a power ball 60
2. Choose 5 numbers . . . . $\binom{59}{5}$

$60 \cdot \binom{59}{5}$

Approach 2:

1. Choose 5 numbers . . . . $\binom{60}{5}$
2. Choose a power ball 55

$\binom{60}{5} \cdot 55$

We conclude that $60 \cdot \binom{59}{5} = \binom{60}{5} \cdot 55$

In general $n\binom{n-1}{k} = \binom{n}{k}(n-k)$

Proof:

$$n \cdot \frac{(n-1)!}{k!(n-1-k)!} = \frac{n!(n-k)}{k!(n-k)!} = \binom{n}{k}(n-k)$$

6. Consider the set $S = \{1, 2, 3, \ldots, n\}$. How many subsets of S contain 1? How many subsets of S contain 1 and have size k?

   Solution

   $$S = \{1, 2, 3, \ldots, n\}$$

   # subsets that contain 1: 2^{n-1}

   # subsets of size k that contain 1: $\binom{n-1}{k-1}$
   (choose $k-1$ elements from $\{2, 3, \ldots, n\}$)

7. How many words of length 7 can you make using the alphabet $\{a, b, c \ldots, z]$ (words don't have to be in the dictionary).

   Solution

   Using the product rule, we have:

   $$\underbrace{26 \cdot 26 \cdot \ldots \cdot 26}_{7 \text{ times}} = 26^7 \text{ (Product rule)}$$

   Also, you would obtain the same answer if you think in terms of the selection framework: choose 7 out of 26 with order and repetition.

Saad Mneimneh (c) Copyright Material - Illegal To Post

8. Consider the number of ways we can seat n people on k different chairs. Is this number greater than or less than the number of ways we can seat k people on n different chairs? Justify your answer.

   Solution

   Both scenarios have the same count. It's just language. Whether you seat n people on k chairs or you place n chairs on top of k people, it's the same mathematically. The answer is always $P_{\min(n,k)}^{\max(n,k)}$. It's a $\min(n, k)$-permutation of $\max(n, k)$ things.

Problem

(a) Assume you have n objects numbered $1, 2, \ldots, n$. Call a good permutation one that places object 1 before object n (otherwise, call the permutation bad). How many good permutations are there? You must count the good permutations by following the approach outlined below (and the product rule). Simplify your answer as much as possible after applying the product rule (you should obtain something surprisingly simple and interesting, hopefully...)

Solution

	# of ways
1. Choose two positions out of n position	$\binom{n}{2}$
2. Place objects 1 and n in these 2 position	1
3. Permute the other objects using remaining positions	$(n-2)!$

By the product rule, we have:

$$\binom{n}{2} \cdot 1 \cdot (n-2)! = \frac{n!}{2!(n-2)!} \cdot (n-2)! = \frac{n(n-1)(n-2)!}{2} = \frac{n!}{2}$$

So exactly half of the permutations are good.

(b) Use what you have discovered in part (a) to solve the following: Imagine you have a pair of gloves, a pair of socks, and a pair of boots. These are labeled GL, GR, SL, SR, BL, BR, where the first letter indicates glove, sock, or boot, and the second letter indicates left or right. In how many ways can

Saad Mneimneh (c) Copyright Material - Illegal To Post

you put these on?

Hint: We are looking at permutations of 6 objects here, but SL must be put on before BL, and SR must be put on before BR.

Solution

Given the set {GL, GR, SL, SR, BL, BR}, We wish to find all permutations of where the elements' appearance in the set determine the order in which we put on the garments. In this regard, the order of the elements maters and presents a challenge: we cannot put a boot on a foot that does not yet have a sock. Of the 6! total permutations of this set corresponding with the total number of ways to put the garments, only some of these permutations are "good" for this reason.

Let's consider the good permutations where SL occurs before BL. How many are there? Exactly half of the total number of permutations, as we discovered in part (a). So we should have $\frac{6!}{2} = \frac{720}{2} = 360$ permutations that are good for SL and BL. By the same argument, among those permutations, only half of them are good for SR and BR. So the total number of good permutations is $\frac{360}{2} = 180$ permutations.

Saad Mneimneh (c) Copyright Material - Illegal To Post