CSCI 150 Discrete Mathematics
Homework 3

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Exercises

1. Find $(\{2,4,6\} \cup \{6,4\}) \cap \{4,6,8\}$
2. Find $\{1,2,3\} \times \{0,1\}$
3. Find $P(\{x,y,z\}) - P(\{x,z\})$, where $P(S)$ is the power set of S (the set of all subsets of S), and $S - T = \{x | x \in S \text{ and } x \notin T\}$.
4. Find $\mathcal{P}(\mathcal{P}(\{\mathcal{P}\}))$
5. Describe the set of all 3-digit positive integers using set notation.
6. We have seen in class that the number of subsets of S when $|S| = n$ is 2^n, and is also given by

$$\sum_{k=0}^{n} \binom{n}{k} = \binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{n}$$

   What does the following mean in English in the context of set theory?

$$\text{"If } n > 0\text{, then } \sum_{k \text{ is even}} \binom{n}{k} = \sum_{k \text{ is odd}} \binom{n}{k}\text{"}$$

   We assume that $\binom{n}{k} = 0$ if $k > n$.

7. Let $S = \{1, 2, \ldots, n\}$. How many subsets of S have size k and do not have $\{1, 2\}$ as a subset?

8. Which of the following is true? (there is only one): $\emptyset = \{\emptyset\}$, $|\emptyset| = 0$, $|P(\emptyset)| = 0$, $\emptyset \in \{\ \}$.

9. Let P be a set of people and U a set of umbrellas. Define $f : P \to U$ such that $f(p) = u$ means person p owns umbrella u (each person owns exactly one umbrella). Transform the following statements into an English meaning about people and umbrellas:

   - f is one-to-one
   - f is not onto
   - f is a bijection

10. Let R be the set of the rainy days of the year and assume that every rainy day $r \in R$ can be either good or bad. For the people and umbrellas above, express the following English statement using the symbols in $\{\forall, \exists, \Rightarrow\}$.

    If two people share the same umbrella, then all rainy days are bad

11. Consider the following 4 functions.

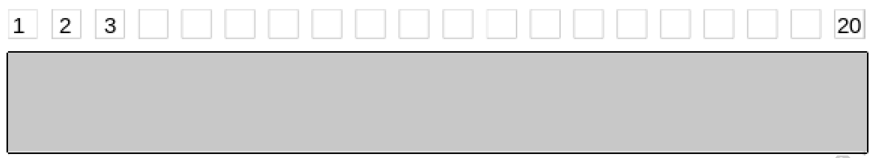
    - $f : \mathbb{N} \to \mathbb{R}$, $f(x) = 1/x$.
    - $g : \mathbb{N} \times \mathbb{N} \to \mathbb{Q}^+$, $g(x, y) = x/y$, where $\mathbb{Q}^+$ is the set of all rational numbers greater than zero (the positive rational numbers).
    - $h : \mathbb{Z} \to \mathbb{Z}$, $h(x) = x^2$.
    - $w : \mathbb{R} \to \mathbb{R}$, $w(x) = 3x + 1$.

    Place them appropriately in the following four categories:

		one-to-one	
		No	Yes
onto	No	?	?
	Yes	?	?

Problem

A table with 20 chairs is seating 20 people as shown below.



This problem is concerned with the number of ways we can select two people out of the 20 with the condition that they are not neighbors, i.e. they are not sitting next to each other.

(a) **Addition rule**: Find the number of non-neighboring pairs by first finding all ways of selecting two people, then subtract from that all pairs consisting of two neighbors. How is that the addition rule?

(b) **Handshake Lemma**: Construct a graph where each person is a vertex, and two vertices are connected by an edge when their corresponding people are **not** neighbors. Using the handshake lemma, count the number of edges (which is by construction the same as the number of non-neighboring pairs). *Note*: You don't have to draw the graph, just reason about degrees of vertices.

(c) **Bijection**: Let S be the set defined below.

$$S = \{(i,j) \in \mathbb{N}^2 \;\Big|\; i \leq 20, j \leq 20, j - i \geq 2\}$$

Convince yourself that the elements of S represent non-adjacent pairs (do you see why?). Rather than finding $|S|$, we are going to define a bijection $f : S \to T$ and find $|T|$ instead. This works because the existence of a bijection means $|S| = |T|$.

$$T = \{(x,y,z) \in \mathbb{N}^3 \;\Big|\; x \geq 1, y \geq 2, z \geq 0, x + y + z = 20\}$$

Consider the function $f : S \to T$, where

$$f(i,j) = (x,y,z) = (i, j - i, 20 - j)$$

To show that f is a function, observe that

$$x = i \geq 1$$

$$y = j - i \geq 2$$

$$z = 20 - j \geq 0$$

$$x + y + z = i + (j - i) + (20 - j) = 20$$

So f maps every element $(i, j) \in S$ to exactly one element of T. To list a few examples, $f(3, 7) = (3, 4, 13)$, $f(12, 14) = (12, 2, 6)$, and $f(1, 20) = (1, 19, 0)$.

1) Show that f is one-to-one and onto. Recall that to show that f is one-to-one, we need to show that if $f(i_1, j_1) = f(i_2, j_2)$, then $(i_1, j_1) = (i_2, j_2)$.

2) Show that f is onto. Recall that to show that f is onto, for a given $(x, y, z) \in T$, we need to find $(i, j) \in S$ such that $f(i, j) = (x, y, z)$.

3) Find $|T|$ by counting the number of integer solutions to $x + y + z = 20$ given the constraints. *Note*: If we have not seen how to find this, we will in the next lecture.

(d) **Sum notation**: Consider the following program.

```
s = 0
for i in range(1, 19)
    t = 0
    for j in range(i + 2, 21)
        t = t + 1
        s = s + 1
    print(t)
print(s)
```

Implement this program in any language you like and observe the output. Can you see why this program computes the number of non-neighboring pairs? Convert the program into a nested $\sum$ notation and compute it.