CSCI 150 Discrete Mathematics
Homework 3

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Solution

Exercises

1. Find $(\{2,4,6\} \cup \{6,4\}) \cap \{4,6,8\}$

   Solution

   $(\{2,4,6\} \cup \{6,4\}) \cap \{4,6,8\} = \{2,4,6\} \cap \{4,6,8\} = \{4,6\}$

2. Find $\{1,2,3\} \times \{0,1\}$

   Solution

   $$\{1,2,3\} \times \{0,1\} = \{(1,0),(1,1),(2,0),(2,1),(3,0),(3,1)\}$$

3. Find $\mathcal{P}(\{x,y,z\}) - \mathcal{P}(\{x,z\})$, where $\mathcal{P}(S)$ is the power set of S (the set of all subsets of S), and $S - T = \{x | x \in S \text{ and } x \notin T\}$.

   Solution

   $$\mathcal{P}(\{x,y,z\}) = \{\emptyset,\{x\},\{y\},\{z\},\{x,y\},\{x,z\},\{y,z\},\{x,y,z\}\}$$
   $$\mathcal{P}(\{x,z\}) = \{\emptyset,\{x\},\{z\},\{x,z\}\}$$

   Now, listing all elements in the first set that are not elements of the second:

   $$\{\{y\},\{x,y\},\{y,z\},\{x,y,z\}\}$$

4. Find $\mathcal{P}(\mathcal{P}(\{\mathcal{P}\}))$

   Solution

   $\mathcal{P}(\{\mathcal{P}\}) = \{\emptyset, \{\mathcal{P}\}\}$

   $\mathcal{P}(\mathcal{P}(\{P\})) = \{\emptyset, \{\emptyset\}, \{\{\mathcal{P}\}\}, \{\emptyset, \{\mathcal{P}\}\}\}$

5. Describe the set of all 3-digit positive integers using set notation.

   Solution

   Here's one way:
$$\{x \in \mathbb{N} \mid 99 < x < 1000\}$$

6. We have seen in class that the number of subsets of S when $|S| = n$ is 2^n, and is also given by
$$\sum_{k=0}^{n} \binom{n}{k} = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n}$$
   What does the following mean in English in the context of set theory?
$$\text{“If } n > 0\text{, then } \sum_{k \text{ is even}} \binom{n}{k} = \sum_{k \text{ is odd}} \binom{n}{k}\text{”}$$
   We assume that $\binom{n}{k} = 0$ if $k > n$.

   Solution

   “If a set S is not empty ($|S| > 0$), then the number of subsets of S with even size is equal to the number of subsets of S with odd size.”

   Observe that the above statement is not true if S is empty, since in that case S has only sets of even size (the empty set).

7. Let $S = \{1, 2, \ldots, n\}$. How many subsets of S have size k and do not have $\{1, 2\}$ as a subset?

   Solution

   The number of subsets of size k that do not have 1 and 2 as elements is equal to $\binom{n-2}{k}$, since we need to select all k elements from $\{3, 4, \ldots, n\}$. The number of subsets of size k that contain 1 as element, but not 2, is $\binom{n-2}{k-1}$, since we need to select $k - 1$ elements other than 1 from $3, 4, \ldots, n\}$. Similarly, the number of subsets of size k that contain 2 as element, but not 1, is $\binom{n-2}{k-1}$. By the addition rule (these three categories of subsets are disjoint), the number of subsets of size k that do not include $\{1, 2\}$ as a subset is:

$$\binom{n-2}{k} + 2\binom{n-2}{k-1}$$

If $k \geq 2$, one could have also solved this differently, by first finding the number of subsets of size k that include $\{1, 2\}$ as a subset, and subtracting that from the total number of sets of size k. The number of subsets of size k that include $\{1, 2\}$ as subset is $\binom{n-2}{k-2}$. Therefore, the answer is:

$$\binom{n}{k} - \binom{n-2}{k-2}$$

Both answers should be equal, and that's the power of counting. It helps us establish equalities that are not trivial to conceive algebraically. In other words, we can write:

$$\binom{n-2}{k} + 2\binom{n-2}{k-1} + \binom{n-2}{k-2} = \binom{n}{k}, \quad 2 \leq k \leq n-2$$

8. Which of the following is true? (there is only one): $\emptyset = \{\emptyset\}$, $|\emptyset| = 0$, $|P(\emptyset)| = 0$, $\emptyset \in \{\ \}$.

   Solution

   $|\emptyset| = 0$ is true. Everything else above is false.

9. Let P be a set of people and U a set of umbrellas. Define $f : P \to U$ such that $f(p) = u$ means person p owns umbrella u (each person owns exactly one umbrella). Transform the following statements into an English meaning about people and umbrellas:

   - f is one-to-one
   - f is not onto
   - f is a bijection

   Solution

   - f is one-to-one: No two people own the same umbrella
   - f is not onto: Some umbrellas are not owned
   - f is a bijection: Each umbrella is owned by exactly one person

10. Let R be the set of the rainy days of the year and assume that every rainy day $r \in R$ can be either good or bad. For the people and umbrellas above, express the following English statement using the symbols in $\{\forall, \exists, \Rightarrow\}$.

   If two people share the same umbrella, then all rainy days are bad

   Solution

$$(\exists u, v \in P, f(u) = f(v) \wedge u \neq v) \implies \forall r \in R, r \text{ is bad}$$

11. Consider the following 4 functions.

   - $f : \mathbb{N} \to \mathbb{R}$, $f(x) = 1/x$.
   - $g : \mathbb{N} \times \mathbb{N} \to \mathbb{Q}^+$, $g(x, y) = x/y$, where $\mathbb{Q}^+$ is the set of all rational numbers greater than zero (the positive rational numbers).
   - $h : \mathbb{Z} \to \mathbb{Z}$, $h(x) = x^2$.
   - $w : \mathbb{R} \to \mathbb{R}$, $w(x) = 3x + 1$.

   Place them appropriately in the following four categories:

		one-to-one	
		No	Yes
onto	No	?	?
	Yes	?	?

Solution

- $f : \mathbb{N} \to \mathbb{R}$, $f(x) = 1/x$.
  - Onto:
    Recognize that $1/x$ will always be a rational number. Since there are numbers that belong to the real number but are not rational (e.g. π, e, $\sqrt{2}$ etc.) this function cannot be onto since the function only produces rational numbers.

- Ono-to-one:

  We can prove the function is one-to-one by showing that if $f(x_1) = f(x_2)$, then $x_1 = x_2$. This is obvious: $1/x_1 = 1/x_2$ means $x_1 = x_2$.

- $g : \mathbb{N} \times \mathbb{N} \to \mathbb{Q}^+$, $g(x, y) = x/y$, where $\mathbb{Q}^+$ is the set of all rational numbers greater than zero (the positive rational numbers)

  - Onto:

    Recalling the definition of a rational number.... So any positive rational number a/b can be expressed as $g(a, b)$. Therefor, g is onto.

  - One-to-one:

    An analysis of the function reveals that the function produces a rational number from two natural numbers. Since there are multiple ways to represent the same rational number, there should be a way to arrive to the same value in Q^+. For example, $g(1, 2) = g(2, 4) = 1/2$. So the function is not one-to-one.

- $h : \mathbb{Z} \to \mathbb{Z}$, $h(x) = x^2$.

  - Onto:

    We can describe the behavior of this function as squaring. Therefore, not every number in $y \in \mathbb{Z}$ corresponds to an $x \in \mathbb{Z}$, such that $y = x^2$. For instance, there is no $x \in \mathbb{Z}$ such that $h(x) = 3$. So the function is not onto.

  - One-to-one:

    The function is also not one-to-one; for instance, $h(-2) = h(2) = 4$.

- $w : \mathbb{R} \to \mathbb{R}$, $w(x) = 3x + 1$.

  - Onto:

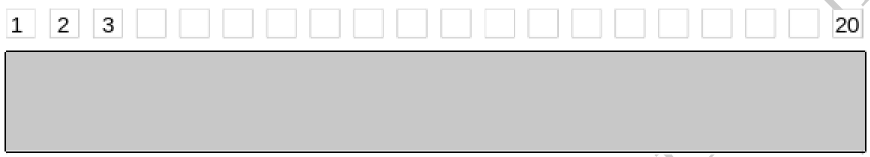
    Consider $y \in \mathbb{R}$. We can find $x \in \mathbb{R}$ such that $w(x) = y$. All we need is $3x + 1 = y$; therefore, $x = (y - 1)/3$, which is obviously in $\mathbb{R}$.

- One-to-one:

  Observe that $w(x_1) = w(x_2) \Rightarrow 3x_1 + 1 = 3x_2 + 1 \Rightarrow x_1 = x_2$. Therefore, w is one-to-one. This makes w a bijection, since it is both onto and one-to-one.

Problem

A table with 20 chairs is seating 20 people as shown below.



This problem is concerned with the number of ways we can select two people out of the 20 with the condition that they are not neighbors, i.e. they are not sitting next to each other.

(a) **Addition rule**: Find the number of non-neighboring pairs by first finding all ways of selecting two people, then subtract from that all pairs consisting of two neighbors. How is that the addition rule?

Solution

There are $\binom{20}{2}$ pairs. There are 19 neighboring pairs $(1,2), (2,3), \ldots, (19,20)$. By the addition rule, the total number of pairs must be equal to the sum of two numbers: the number of neighboring pairs and the number of non-neighboring pairs. Therefore, the number of non-neighboring pairs must be $\binom{20}{2} - 19 = \frac{20 \cdot 19}{2} - 19 = \frac{20 \cdot 19 - 2 \cdot 19}{2} = \frac{19 \cdot 18}{2}$.

(b) **Handshake Lemma**: Construct a graph where each person is a vertex, and two vertices are connected by an edge when their corresponding people are **not** neighbors. Using the handshake lemma, count the number of edges (which is by construction the same as the number of non-neighboring pairs). *Note*: You don't have to draw the graph, just reason about degrees of vertices.

Solution

In constructing the suggested graph, we observe that two of the vertices (vertex 1 and vertex 20) have degree 18, and the rest of the vertices (18 of them) have degree 17. Therefore, the sum of all degrees is $2 \times 18 + 18 \times 17 = 18 \times 19$.

By the Handshake Lemma, the number of edges must be hald of that, which is $\frac{19\times 18}{2}$.

(c) **Bijection**: Let S be the set defined below.

$$S = \{(i,j) \in \mathbb{N}^2 \mid i \leq 20, j \leq 20, j - i \geq 2\}$$

Convince yourself that the elements of S represent non-adjacent pairs (do you see why?). Rather than finding $|S|$, we are going to define a bijection $f : S \to T$ and find $|T|$ instead. This works because the existence of a bijection means $|S| = |T|$.

$$T = \{(x,y,z) \in \mathbb{N}^3 \mid x \geq 1, y \geq 2, z \geq 0, x + y + z = 20\}$$

Consider the function $f : S \to T$, where

$$f(i,j) = (x,y,z) = (i, j - i, 20 - j)$$

To show that f is a function, observe that

$$x = i \geq 1$$

$$y = j - i \geq 2$$

$$z = 20 - j \geq 0$$

$$x + y + z = i + (j - i) + (20 - j) = 20$$

So f maps every element $(i,j) \in S$ to exactly one element of T. To list a few examples, $f(3,7) = (3,4,13)$, $f(12,14) = (12,2,6)$, and $f(1,20) = (1,19,0)$.

1) Show that f is one-to-one. Recall that to show that f is one to one, we need to show that if $f(i_1,j_1) = f(i_2,j_2)$, then $(i_1,j_1) = (i_2,j_2)$.

2) Show that f is onto. Recall that to show that f is onto, for a given $(z,y,z) \in T$, we need to find $(i,j) \in S$ such that $f(i,j) = (x,y,z)$.

3) Find $|T|$ by counting the number of integer solutions to $x + y + z = 20$ given the constraints.

Solution First, let us show that f is one-to-one. Say that $f(i_1,j_1) = f(i_2,j_2)$. This means

$$(i_1, j_1 - i_i, 20 - j_1) = (i_2, j_2 - i_2, 20 - j_2)$$

This means

$$i_1 = i_2$$

$$20 - j_1 = 20 - j_2$$

Therefore, $i_1 = i_2$ and $j_1 = j_2$. This means $(i_1, j_1) = (i_2, j_2)$.

Next, let us show that f is onto. Given $(x, y, z) \in T$, let us find $(i, j) \in S$ such that $f(i, j) = (x, y, z)$. Consider $(i, j) = (x, x + y)$.

$$f(i, j) = f(x, x + y) = (x, (x + y) - x, 20 - (x + y)) = (x, y, z)$$

Now, we need to show that (i, j) is indeed an element of S. Since $x \geq 1$, $y \geq 2$, and $x + y \leq 20$, we know that both x and $x + y$ are elements of $\mathbb{N}$ and at most 20. Furthermore, $j - i = (x + y) - x = y \geq 2$. Therefore, all conditions are satisfied and $(i, j) \in S$.

Finally, let's count:

$$x + y + z = 20$$

$$x \geq 1$$

$$y \geq 2$$

$$x \geq 0$$

By letting $x' = x + 1$ and $y' = y + 2$, we rewrite $x' + y' + z = 20 - 1 - 2 = 17$, where $x', y', z \geq 0$. The number of integer solutions for this equation is $\binom{3+17-1}{17} = \binom{19}{17} = \binom{19}{2} = \frac{19 \cdot 18}{2}$.

(d) **Sum notation**: Consider the following program.

```
s = 0
for i in range(1, 19)
    t = 0
    for j in range(i + 2, 21)
        t = t + 1
        s = s + 1
    print(t)
print(s)
```

Implement this program in any language you like and observe the output. Can you see why this program computes the number of non-neighboring pairs? Convert the program into a nested $\sum$ notation and compute it.

Solution

$$\sum_{i=1}^{18} \sum_{j=i+2}^{20} 1 = \sum_{i=1}^{18}(20-(i+2)+1) = \sum_{i=1}^{18}(19-i) = 18+17+\ldots+1 = \frac{19 \cdot 18}{2}$$