CSCI 150 Discrete Mathematics
Homework 4

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Solution

Exercises

1. You have 4 identical balls, and you want to place them in 10 different containers. In how many ways can we do that if we need to satisfy the condition that at least one container must have at least 2 balls? *Hint*: Solve this using these steps:

   - First, count the possibilities that do not satisfy this condition (i.e. every container has at most 1 ball).
   - Then, count the total number of ways of placing 4 identical balls into 10 different containers.
   - Finally, subtract one from the other.

   Solution
   First, there are $\binom{10}{4}$ possible ways of placing the 4 balls in the 10 containers in such a way that each container gets at most 1 ball. This is simply choosing 4 containers out of the 10 to place a ball in each. Second, to count all possibilities of assigning the balls to the containers, we can consider the number of integer solutions to the equation (it's just a matter of how many times each container is chosen):

$$x_1 + x_2 + \ldots + x_{10} = 4$$

where $x_i \geq 0$. This is $\binom{10+4-1}{10-1} = \binom{13}{9}$. Finally, the answer is $\binom{13}{9} - \binom{10}{4} = 715 - 210 = 505$.

2. You have 10 candies, 2 of which are identical. You want to eat one after breakfast, one after lunch, and one after dinner (childish I know...) Assume that this order of eating the candies is important. In how many ways can you eat 3 candies? *Hint*: Solve this using these steps:

   - First, count the possibilities in which all candies are different (so work with 9 candies).
   - Then, count the possibilities in which 2 of the candies are identical.
   - Finally, add them up.

   Solution:

   First, we choose 3 candies out of 9 different candies with order. This can be done $9!/(9-3)! = 504$. Second, given the 2 identical candies, we have to select one out of the remaining 8 to make 3 candies. This can be done in 8 ways of course. But we still have to permute the 3 candies, and since two of them are identical, this can be done in $3!/(2!1!) = 3$ ways (see anagrams in Note 2; for instance, the number of words we can make using the letters $\{A, A, B\}$). Another way of thinking about it is that the different candy can be the first, the second, or the third to be eaten. By the product rule, we get 24. Finally, we add the two, we get $504 + 24 = 528$.

3. How many integer solutions are there for $x + y + z + w = 15$ if $x \geq 3$, $y > -2$, $z \geq 1$, $w > -3$? (make sure you take care of $\geq$ and $>$).

   Solution

   Let $x = 3 + x'$ where $x' \geq 0$, similarly, $y = -1 + y'$, $z = 1 + z'$, and $w = -2 + w'$, where $y', z', w' \geq 0$. Rewriting the equation, we have:

$$3 + x' - 1 + y' + 1 + z' - 2 + w = 15$$
$$x' + y' + z' + w' = 14$$

   which by standard method has $\binom{4-1+14}{4-1} = 680$ solutions.

4. There are 14 men and 9 women. They are to be seated on 23 chairs in a row such that no two women sit next to each other. How many ways are possible? (*Hint*: we did something similar with bits).

Solution

This is very similar to the binary words with no consecutive 1s. The 9 women divide the men into 10 groups. The group on the far left, and the group on the far right may be empty. But all the middle groups must have at least one man. This leads to the equation:

$$x_1 + \ldots + x_{10} = 14$$

with $x_1, x_{10} \geq 0$ and $x_2, \ldots, x_9 \geq 1$. We get

$$x_1 + (1 + y_2) + \ldots + (1 + y_9) + x_{10} = 14$$

$$x_1 + y_2 + \ldots + y_9 + x_{10} = 6$$

with all non-negative integers. The solution is $\binom{10-1+6}{10-1} = \binom{15}{9}$.

BUT, the people are not bits! We can still permute the men in 14! ways, and the women in 9! ways. The final answer is $14!9!\binom{15}{9}$.

5. What is the coefficient of x^3y^2 in $(x + y + 2)^{10}$?

Solution

By application of the multinomial theorem: The coefficient of $x^3y^22^5$ is $\binom{10}{3\ 2\ 5} = \frac{10!}{3!2!5!}$. So the coefficient of x^3y^2 is $\frac{10!}{3!2!5!}2^5$. Alternatively, one can apply the binomial theorem twice. Let $z = y+2$. In $(x+z)^{10}$, the coefficient of x^3z^7 is $\binom{10}{7}$. In $z^7 = (y+2)^7$, the coefficient of $y^2 \cdot 2^5$ is $\binom{7}{5}$. So the answer is $\binom{10}{7}\binom{7}{5}2^5$, which is the same as above. Both methods lead to 80, 640.

6. Find

$$\binom{100}{0}\left(\frac{1}{3}\right)^{100}6^0 + \binom{100}{1}\left(\frac{1}{3}\right)^{101}6^1 + \ldots + \binom{100}{100}\left(\frac{1}{3}\right)^{200}6^{100}$$

Hint: This form is not the binomial theorem yet!

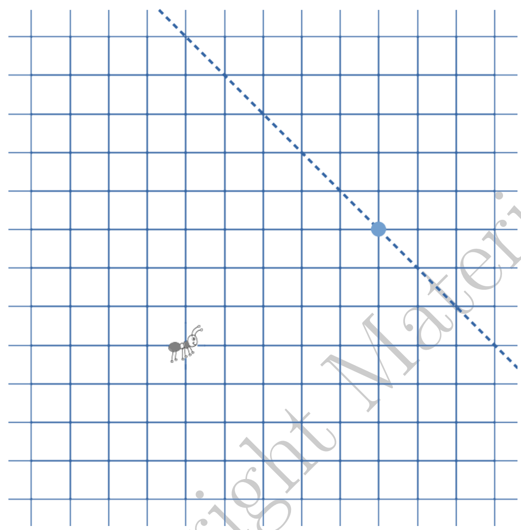
Solution

$$\frac{1}{3^{200}}\left[\binom{100}{0}3^{100}6^0 + \binom{100}{1}3^{99}6^1 + \ldots + \binom{100}{100}3^06^{100}\right]$$

$$= \frac{1}{3^{200}}(3+6)^{100} = \frac{1}{3^{200}}9^{100} = \frac{1}{3^{200}}3^{200} = 1$$

Problem

An ant sits on an infinite grid at $(0,0)$ as shown below. There is also food indicated by a dot at $(5,3)$. The ant is not aware yet of the oblique line (that's Broadway of the ant city), but she will be in part (e).



(a) Assume that the ant senses where the food is and will start moving towards it by only making UP and RIGHT moves. How many possible paths are there that will bring the ant to the food? *Hint*: think of a bijection from the set of paths to some special set of binary words (what is that set?).

Solution

Each path must consist of 5 RIGHT moves and 3 UP moves. So a path can be thought of as an anagram with 5 Rs and 3 Us. Therefore, the number of possible paths is $\binom{8}{3} = \binom{8}{5} = \frac{8!}{5!3!}$.

Here's another idea: Let P be the set of paths and $B_{8,5}$ be the set of binary words with 8 bits and 5 1s. Consider the function

$$f : P \to B_{8,5}$$

where $f(p)$ is obtained by replacing each R with 1 and each U with 0. It is easy to see that f is both one-to-one and onto (try to prove both). Therefore, f is a bijection and $|P| = |B_{8,5}| = \binom{8}{5}$.

In fact, representing a path as an anagram above is effectively making a bijection between the set of paths and anagrams of RRRRRUUU.

(b) In this version of the story, the ant has no idea where the food is. So she started to move from $(0, 0)$ by making a sequence of random UP, DOWN, LEFT, and RIGHT moves, only to find herself back to her original position after 8 moves. Although she was disappointed, she thought: "How many possible paths of length 8 will bring me back to my original position?" Well, she did not really think that, but I did. Please count those using the following strategy:

- First, divide the paths into disjoint categories, depending on the number of UP moves in the path.
- Then observe that the number of UP moves completely determines the number of all other types of moves!
- Then think of the path as an anagram of length 8 made of U, D, L, and R.
- Finally, use the addition rule.

Solution

#UP	#DOWN	#LEFT	#RIGHT	#path
0	0	4	4	8!/(0!0!4!4!)
1	1	3	3	8!/(1!1!3!3!)
2	2	2	2	8!/(2!2!2!2!)
3	3	1	1	8!/(3!3!1!1!)
4	4	0	0	8!/(4!4!0!0!)

The total # path is 4900.

In general, the number of length $2n$ paths that bring the ant back to the origin is

$$\sum_{k=0}^{n} \frac{(2n)!}{k!k!(n-k)!(n-k)!} = \sum_{k=0}^{n} \frac{(2n)!}{n!n1} \frac{n!}{k!(n-k)!} \frac{n!}{k!(n-k)!}$$

$$= \frac{(2n)!}{n!n!} \sum_{k=0}^{n} \binom{n}{k}\binom{n}{k} = \binom{2n}{n} \underbrace{\sum_{k=0}^{n} \binom{n}{k}\binom{n}{n-k}}_{\text{\# of ways to choose n from 2n}} = \binom{2n}{n}^2$$

So when $n = 4$, we get $\binom{8}{4}^2 = 70^2 = 4900$.

(c) The ant gained some magical powers (and let's not get into how this happened)! Now she can move according to the following function:

$$f : \mathbb{Z}^2 \to \mathbb{Z}^2$$

$$f(i,j) = (k,l) = (i+j, i-j)$$

So if the ant is at position (i,j), she will next be at position $(k,l) = f(i,j)$, and so on.

- Give convincing arguments that f is one-to-one but **not** onto. To show one-to-one, use our standard strategy by starting with $f(i,j) = f(i',j')$ and concluding that $(i,j) = (i',j')$.
- Show that despite her magical powers, there is only one starting position, except position $(5,3)$ itself, that will eventually bring the ant to the food. *Hint*: think backwards.

Solution
Let's prove one-to-one:

$$\begin{aligned}
f(i_1,j_1) = f(i_2,j_2) &\implies (i_1+j_1, i_1-j_1) = (i_2+j_2, i_2-j_2) \\
&\implies i_1+j_1 = i_2+j_2 \wedge i_1-j_1 = i_2-j_2 \\
&\implies 2i_1 = 2i_2 \\
&\implies i_1 = i_2 \\
i_1+j_1 = i_2+j_2 \wedge i_1 = i_2 &\implies i_1+j_1 = i_1+j_2 \\
&\implies j_1 = j_2
\end{aligned}$$

Therefore $(i_1,j_1) = (i_2,j_2)$, and f is one-to-one.

Observe that if $f(i,j) = (k,l)$ then $k+l = (i+j)+(i-j) = 2i$ thus any pair (k,l) whose sum $k+l$ is odd has no pre-image under f, so f is not onto. The pre-image of (5,3) is (4,1) i.e. $f(4,1) = (5,3)$, and because $4+1 = 5$ is odd, there are no other points which will bring her to (5,3).

(d) Aimless, the ant decided that she will make a total of 8 moves with at least one UP move and at least one RIGHT move. She also decided that she will first make all the UP moves, then all the LEFT moves, then all the

DOWN moves, then all the RIGHT moves. How many paths are possible?

Solution
What matters is how many times each direction is chosen! Let's use four variables and find integer solutions.
$r + u + d + l = 8$
  $r \geq 1 \to r = r' + 1$
  $u \geq 1 \to u = u' + 1$
  $l \geq 0$
  $d \geq 0$
$r' + u' + d + l = 6 \quad (n = 4, k = 6)$
#paths $= \binom{6+4-1}{4-1} = \binom{9}{3}$

(e) If the ant will only make UP and RIGHT moves, but will also decide whether to deposit pheromones every time she makes a RIGHT move, in how many ways (including pheromone pattern) can she reach Broadway (the oblique line)? *Hint 1*: Think about the number of ways to reach each of the grid points on the oblique line, then use the addition rule. *Hint 2*: The Binomial theorem will be helpful here.

Solution
There are $\binom{8}{k}$ paths that bring the ant to point $(k, 8-k)$, for $k = 0 \ldots 8$. These paths have k RIGHT moves, which means there are 2^k ways to deposit pheromones along each such path. By the addition rule, we have:

$$\binom{8}{0}2^0 + \binom{8}{1}2^1 + \binom{8}{2}2^2 + \ldots + \binom{8}{8}2^8$$

Using the Binomial Theorem, this is $(1+2)^8 = 3^8$.