CSCI 150 Discrete Mathematics
Homework 5

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Solution

1. Think of the most fascinating theorem that you have ever encountered. It could be a simple one, or it could be very sophisticated. Whatever the statement of that theorem is, let's call it P. Consider the following proposition:

$$\text{pigs can fly} \Rightarrow P$$

   - Why is the above proposition true?
   - How come this is not a proof of P itself?

Solution

The proposition is true since "pigs can fly" is false. Recall that both $0 \Rightarrow 0$ and $0 \Rightarrow 1$ are true propositions. Therefore, P cannot be determined to be true or false from the fact that $0 \Rightarrow P$ is true.

2. For each of the following, find whether it is always true, always false, or neither:

$$0 \wedge P$$

$$1 \wedge P$$

$$0 \vee P$$

$$1 \vee P$$

$$0 \Rightarrow P$$

$$1 \Rightarrow P$$

$$P \vee \neg P$$

$$P \wedge \neg P$$

$$P \Rightarrow \neg P$$

Solution

Always true: $1 \vee P$, $0 \Rightarrow P$, $P \vee \neg P$.
Always false: $0 \wedge P$, $P \wedge \neg P$.
Neither: the rest

3. Let P and Q be two propositions and consider the following proposition:

$$(P \Rightarrow Q) \vee (Q \Rightarrow P)$$

Show that the above proposition is always true:

- by means of a truth table:

  Solution

P Q	$P \Rightarrow Q$	$Q \Rightarrow P$	$(P \Rightarrow Q) \vee (Q \Rightarrow P)$
0 0	1	1	1
0 1	1	0	1
1 0	0	1	1
1 1	1	1	1

- by replacing any proposition of the form $X \Rightarrow Y$ by $\neg X \vee Y$ and using the associative property of $\vee$.

$$(P \Rightarrow Q) \vee (Q \Rightarrow P)$$
$$(\neg P \vee Q) \vee (\neg Q \vee P)$$
$$\neg P \vee (Q \vee \neg Q) \vee P$$
$$\neg P \vee 1 \vee P$$
$$(\neg P \vee 1) \vee P$$
$$1 \vee P$$
$$1$$

4. An integer n is a multiple of 3 if $n = 3k$, and is not a multiple of 3 if $n = 3k+1$ or $n = 3k+2$ (these are all possible cases), where $k \in \mathbb{Z}$. Consider the following statement:

$$\forall n \in \mathbb{Z},\ (n^2 \text{ is not a multiple of } 3 \Rightarrow n \text{ is not a multiple of } 3)$$

(a) Prove the statement is true by working with its contrapositive.

(b) By finding n^2 for each possible case listed above for n, conclude a stronger result:

$$\forall n \in \mathbb{Z},\ (n \text{ is a multiple of } 3 \Leftrightarrow n^2 \text{ is a multiple of } 3)$$

Solution

(a) The contrapositive: $\underbrace{n \text{ is a multiple of } 3}_{\neg Q} \Rightarrow \underbrace{n^2 \text{ is a multiple of } 3}_{\neg P}$

proof: n is a multiple of 3 $\Rightarrow n = 3k$, where $k \in \mathbb{Z} \Rightarrow n^2 = 9k^2 = 3(3k^2) = 3k'$, where $k' \in \mathbb{Z} \Rightarrow n^2$ is a multiple of 3.

(b) Here are all the cases:
$n = 3k$, $n^2 = 9k^2 = 3(3k^2)$
$n = 3k+1$, $n^2 = (3k+1)^2 = 9k^2 + 6k + 1 = 3(3k^2 + 2k) + 1$
$n = 3k+2$, $n^2 = (3k+2)^2 = 9k^2 + 12k + 4 = 9k^2 + 12k + 3 + 1 = 3(3k^2 + 4k + 1) + 1$

Therefore, n is a multiple of 3 iff n^2 is a multiple of 3.

Observe that one could also establish:

$$(n = 3k+1) \vee (n = 3k+2) \Rightarrow n^2 = 3k' + 1$$

which can read: n is not a multiple of 3 $\Rightarrow n^2$ is not a multiple of 3. That's $\neg P \Rightarrow \neg Q$ which is equivalent to $Q \Rightarrow P$.

5. Prove by a counter example that the following statement is false

$$\forall n \in \mathbb{Z},\ (n^2 \text{ is a multiple of } 4 \Rightarrow n \text{ is a multiple of } 4)$$

Solution

Let $n = 2/$ Then $n^2 = 4$, which is a multiple of 4, but n is not.

6. Prove by contradiction that $\sqrt{3} \notin \mathbb{Q}$.

Suppose $\sqrt{3}$ is rational. Then $\sqrt{3} = \frac{a}{b}$ where $a, b \in \mathbb{Z}$ and the fraction is not reducible (i.e. a and b have no common divisors except 1).

$$\sqrt{3} = \frac{a}{b} \Rightarrow 3 = \frac{a^2}{b^2} \Rightarrow a^2 = 3b^2 \Rightarrow a^2 \text{ is a multiple of } 3 \Rightarrow a \text{ is a multiple of } 3$$

$$b^2 = \frac{a^2}{3} \Rightarrow b^2 = \frac{3k \cdot 3k}{3} = 3k^2 \Rightarrow b^2 \text{ is a multiple of } 3 \Rightarrow b \text{ is multiple of } 3$$

Therefore, a and b have 3 as a common divosor and we have reached a contradiction.

7. Consider $(P \wedge Q) \Rightarrow R$ and $P \Rightarrow (R \vee \neg Q)$. Show that they are equivalent:

- by means of a truth table
- by reasoning about when exactly each of them is false (without a truth table).

Solution

PQR	$P \wedge Q$	$(P \wedge Q) \Rightarrow R$	$R \vee \neg Q$	$P \Rightarrow (R \vee \neg Q)$
000	0	1	1	1
001	0	1	1	1
010	0	1	0	1
011	0	1	1	1
100	0	1	1	1
101	0	1	1	1
110	1	0	0	0
111	1	1	1	1

- $(P \wedge Q) \Rightarrow R$ is false only when $(P \wedge Q)$ is true and R is false. So when $P = 1$, $Q = 1$, and $R = 0$.
- $P \Rightarrow (R \vee \neg Q)$ is false only when P is true and $(R \vee \neg Q)$ is false. So when $P = 1$, $Q = 1$, and $R = 0$.

8. Assume that all of the following are true

$$(P \vee Q) \Rightarrow R$$

$$Q \vee R$$

$$R \Rightarrow P$$

Show by contradiction that P is true.

Solution

Assume P is false.

Since $R \Rightarrow P$ is true, then R must be false. Now, since $Q \vee R$ is true, then Q must be true. We now have $P = 0$, $Q = 1$, and $R = 0$. This makes $(P \vee Q) \Rightarrow R$ false, a contradiction.