CSCI 150 Discrete Mathematics
Homework 6

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Solution (before midnight)

We argued in class that there are no proofs by example, an exception being that we may disprove a statement by providing a *counter example*. For instance, to prove that the polynomial $p(n) = n^2 + n + 41$ does not produce a prime number for every integer $n \geq 0$, we may show that $41^2 + 41 + 41$ is not prime.

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1. Prove that the polynomial $n^2 - 79n + 1601$ does not produce a prime number for every integer $n \geq 0$.

   Solution Consider $n = 1601$. Then,

   $$n^2 - 79n + 1601 = 1601^2 - 79 \cdot 1601 + 1601 = 1601(1601 - 79 + 1) = 1601 \cdot 1523$$

   which is not prime.

2. Prove that the sequence given by

   $$a_n = 1 + \prod_{k=1}^{n} p_k$$

where p_k is the k^{th} prime, is not always prime. Here are the first few values (which are prime):

$$3, 7, 31, 211, 2311, \ldots$$

Solution

$$a_6 = 1 + p_1p_2p_3p_4p_5p_6 = 1 + 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 = 30031 = 59 \cdot 509$$

which is not prime.

Another scenario where the use of an example is appropriate is *existential proofs* when we are interested in showing the truth of a statement of the form:

$$\exists n, P(n)$$

For example, prove that there exist a prime number that is even.

$$\exists n, n \text{ is prime and even}$$

In this case, we can simply "construct" an example. For instance, 2 is prime and is even. Done!
Here's another example: Prove that there exists two perfect squares whose sum is a perfect square.

$$\exists x, y, z \in \mathbb{N}, x^2 + y^2 = z^2$$

Similarly, we can construct an example: $9 + 16 = 25$.

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3. The Fibonacci numbers are given by the following sequence:

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$$

   Prove that there is a Fibonacci number that ends in the digit 7.

   Solution

   0 1 1 2 3 5 8 13 21 34 55 89 144 233 377 ...

4. Prove that there exists two irrational numbers x and y such that xy is rational.
   Solution Consider $\sqrt{2}$ and $1/\sqrt{2}$. We know that $\sqrt{2}$ is irrational. In addition, if a number x is irrational, then $1/x$ is irrational as well (we can prove this by the contrapositive). We are done, since $\sqrt{2} \times 1/\sqrt{2} = 1$ is rational.

Sometimes, it is not easy to construct an explicit example, but we can still prove existence. Such proofs are called "non-constructive". Here's an example: Prove that $x^3 + x - 1 = 0$ has a solution.

$$\exists x \in \mathbb{R}, x^3 + x - 1 = 0$$

The function $f(x) = x^3 + x - 1$ is a continuous function, and $f(0) = -1$ and $f(1) = 1$. This means there must be an x, $0 < x < 1$, such that $f(x) = 0$. Observe that we could not construct the solution itself, but we were able to prove that it exists.

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5. Prove that $x^4 - x - 1 = 0$ has more than one solution.

   Solution

   Let $(x) = x^4 - x - 1$. The function $f(x)$ is a continuous function. In addition, $f(-1) = 1$ and $f(0) = -1$. Therefore, f crosses 0 in the interval $(-1, 0)$. So there must be an $-1 < x < 0$ such that $f(x) = 0$. Similarly, $f(1) = -1$ and $f(2) = 13$. So there must be an $1 < x < 2$ such that $f(x) = 0$. Therefore, $x^4 - x - 1 = 0$ has at least two solutions.

6. Prove that there exist two irrational numbers x and y such that x^y is rational. *Hint*: Think of the number

   $$\left(\sqrt{3}^{\sqrt{2}}\right)^{\sqrt{2}}$$

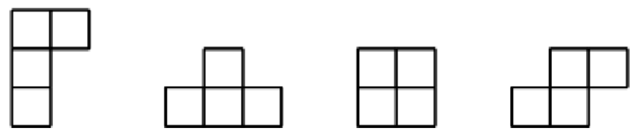
   and consider all possible cases for $\sqrt{3}^{\sqrt{2}}$.

   Solution

   First, observe that the number above is 3. We consider two cases:

   - if $\sqrt{3}^{\sqrt{2}}$ is rational, then let $x = \sqrt{3}$ and $y = \sqrt{2}$. We know that both x and y are irrational, but x^y is rational.
   - if $\sqrt{3}^{\sqrt{2}}$ is irrational, then let $x = \sqrt{3}^{\sqrt{2}}$ and $= \sqrt{2}$. Both x and y are irrational, but $x^y = 3$ is rational.

7. Prove by contradiction that the following tiles cannot be put together to make a perfect square. *Hint*: use a parity argument similar to the one we saw in class.



Solution

A square must have 16 cells and, if we imagine a chessboard pattern, 8 of these cells will be black and 8 will be white. Therefore, there are as many white as black cells. To proceed by contradiction, we assume that the tiles can cover the square exactly. The second tile will have either 3 black cells and 1 white cell, or 3 white cells and 1 black cell. All other tiles will have an equal number of white and black cells. Therefore, the square will have an unequal number of white and black cells, a contradiction.

8. Prove the following using the contrapositive:

$$\forall r \in \mathbb{R} - \{1\}, \frac{r}{r-1} \notin \mathbb{Q} \Rightarrow r \notin \mathbb{Q}$$

Does the statement remain true if we simply reverse the implication?

Solution

The contrapositive is given by:

$$\forall r \in \mathbb{R} - \{1\}, r \in \mathbb{Q} \Rightarrow \frac{r}{r-1} \in \mathbb{Q}$$

Proof: $r \in \mathbb{Q} \Rightarrow r = \frac{a}{b}$ where $a, b \in \mathbb{Z} \Rightarrow \frac{r}{r-1} = \frac{a/b}{a/b-1} = \frac{a}{a-b}$. Since $a - b$ is an integer not equal to 0 ($r \neq 1$ so $a \neq b$), $r/(r-1) \in \mathbb{Q}$.

If we simply reverse the implication, the statement remains true. To show this, here's the statement with the implication reversed:

$$\forall r \in \mathbb{R} - \{1\}, r \notin \mathbb{Q} \Rightarrow \frac{r}{r-1} \notin \mathbb{Q}$$

which is equivalent to the contrapositive:

$$\forall r \in \mathbb{R} - \{1\}, \frac{r}{r-1} \in \mathbb{Q} \Rightarrow r \in \mathbb{Q}$$

which we can prove similarly by starting with $r/(r-1) = a/b$ and concluding $r = a/(a-b)$. Therefore, we can say:

$$\forall r \in \mathbb{R} - \{1\}, \frac{r}{r-1} \notin \mathbb{Q} \Leftrightarrow r \notin \mathbb{Q}$$

9. Prove the following is true:

$$\forall n \in \mathbb{N}, n \text{ is even} \Rightarrow \binom{n}{3} \text{ is even}$$

Hint: If $2x/3$ is an integer, then $x/3$ is an integer because 2 and 3 have no common factors.

Solution

n is even $\Rightarrow n = 2k$, where k is an integer $\Rightarrow \binom{n}{3} = \frac{n(n-1)(n-2)}{3!} = \frac{2k(2k-1)(2k-2)}{3!} = \frac{2[k(2k-1)(k-1)]}{3}$. Since $\binom{n}{3}$ is an integer of the form $2x/3$, it must be that $x/3$ is an integer. Therefore, $\binom{n}{3}$ is even.

10. Which of the following sets is countable and which is uncountable (try your best to explain your answer)?

- The set of all cups on Earth

  Solution This set is finite, so it's countable.

  Note: If by "set of all cups on Earth", we mean all cups that are made and will ever be made, and we assume that humans will always make cups on Earth, and that humans will never perish, and that Earth will never perish, then this set in infinite. However, even then, it's countable, as one could find a bijection with $\mathbb{N}$ by ordering the cups, say by the timestamp of their production (and breaking ties in some way assuming there are only finitely many cups that can share the exact same timestamp).

- The set of all real numbers in (0,1)

  Solution: This is uncountable. Using the same diagonalization method, and given a function $f : \mathbb{N} \to (0,1)$, we can construct a number $x = 0.x_1x_2x_3 \ldots \in (0,1)$, where the i^{th} digit x_i is different from the i^{th} digit of $f(i)$. In fact, one can find a bijection from $(0,1)$ to $\mathbb{R}$.

- The set of all finite binary sequences

  Solution

  This is countable. We can order the sequences by their length. Since there are finitely many sequences of a given length ℓ (exactly 2^ℓ sequences), every sequence will have a finite rank. To see this, consider the total number of sequences with length at most ℓ, we have $\sum_{i=0}^{\ell} 2^i = 2^{\ell+1} - 1$ sequences. Therefore, any sequence of length ℓ has a finite rank.

- The set $\mathbb{R} - \mathbb{Z}$

  Solution

  This is uncountable. Here's a proof by contradiction: Assume $\mathbb{R} - \mathbb{Z}$ is countable. Since the union of two countable sets is countable, then $(\mathbb{R} - \mathbb{Z}) \cup \mathbb{Z} = \mathbb{R}$ is countable, a contradiction.