CSCI 150 Discrete Mathematics
Homework 7

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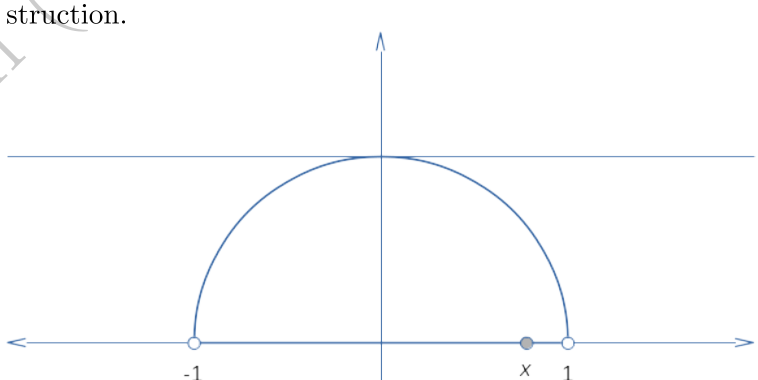
Solution

Note: Some ideas in this homework are taken from the book Gentle Introduction to the Art of Mathematics.

1. (Optional, to understand that the interval $(-1, 1)$ is "as big" as the entire set $\mathbb{R}$.)

   - Find a function $f : (-1, 1) \to \mathbb{R}$ that is both one-to-one and onto. This establishes the claim. *Hint*: design your function such that it's continuous and takes -1 to $-\infty$ and $+1$ to ∞.
   - In this approach, we will map $(-1, 1)$ to $\mathbb{R}$ by a geometric construction.



For any $x \in (-1,1)$:

(a) obtain its vertical projection on the unit circle in the upper half plane, then
(b) make an appropriate projection of that point onto the tangent line (the line $y = 1$), and finally
(c) project vertically onto $\mathbb{R}$ (the line $y = 0$).

Obviously, you only have to figure out (b). Show your work and explain. Extra: can you obtain the function corresponding to this geometric argument?

Extra Challenge: (You are not required to do this) Find a bijection from $[-1,1]$ to $\mathbb{R}$. This time -1 and 1 are elements of the domain.

Solution

- Both of the following functions are bijections from $(-1,1)$ to $\mathbb{R}$:

$$f(x) = \ln \frac{1+x}{1-x}$$

$$g(x) = \tan\left(\frac{\pi}{2}x\right)$$

Take $f(x)$ for instance.
<u>one-to-one</u>:

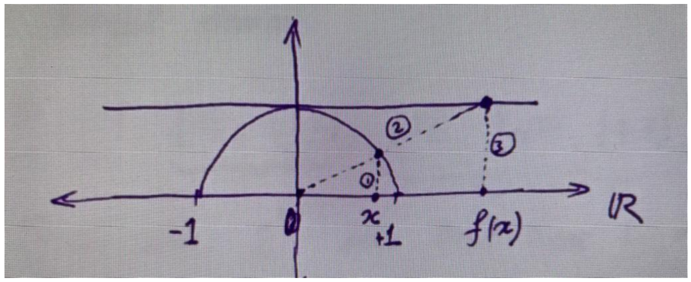
$$f(x_1) = f(x_2) \Rightarrow \ln \frac{1+x_1}{1-x_1} = \ln \frac{1+x_2}{1-x_2} \Rightarrow \frac{1+x_1}{1-x_1} = \frac{1+x_2}{1-x_2} \text{ (log itself is one-to-one)}$$

$$\Rightarrow (1+x_1)(1-x_2) = (1+x_2)(1-x_1) \Rightarrow 1+x_1-x_2-x_1x_2 = 1+x_2-x_1-x_1x_2$$

$$\Rightarrow x_1 - x_2 = x_2 - x_1 \Rightarrow 2x_i = 2x_2 \Rightarrow x_1 = x_2$$

<u>onto</u>: For any $y \in \mathbb{R}$, we can find an x such that $\ln \frac{1+x}{1-x} = y$. Simply solve for x to obtain $x = \frac{w^y-1}{e^y+1}$, and observe that $x \in (-1,1)$ because $e^y \geq 0$.

- Geometric interpretation:
  - Vertical projection on semi-circle
  - Projection on tangent using center of circle
  - Vertical projection on $y = 0$



We can actually obtain the function corresponding to the above geometric construction. First, the height of the point on the semi-cricle can be obtained using Pythagoras theorem: $x^2 + h^2 = 1$. Therefore $h = \sqrt{1 - x^2}$. Second, by similar triangles, we have:

$$\frac{x}{h} = \frac{f(x)}{1}$$

which means $f(x) = x/\sqrt{1 - x^2}$.

- Extra challenge: We can construct a bijection from $[-1, 1]$ to $(-1, 1)$. This means, by function composition, there is a bijection from $[-1, 1]$ to $\mathbb{R}$. Here's the function:

$$f(x) = \begin{cases} 1/2 & x = -1 \\ 1/4 & x = 1 \\ 1/2^{i+2} & x = 1/2^i \\ x & \text{otherwise} \end{cases}$$

So we use the infinite sequence $1/2, 1/4, 1/8, 1/16, 1/32, \ldots$ to map $-1, 1, 1/2, 1/4, 1/8, \ldots$ (shifting). This takes care of -1 and $+1$. Then any x that is not of the form $1/2^i$ will map to itself.

2. We learned in class that for any set A, $|A| < |\mathcal{P}(A)|$. For instance, this means that $\mathcal{P}(\mathbb{N})$, the set of all subsets of $\mathbb{N}$ is uncountable. Let $\mathcal{P}_{\mathcal{F}}(\mathbb{N})$ be the set of all **finite** subsets of $\mathbb{N}$.

$$\mathcal{P}_{\mathcal{F}}(\mathbb{N}) = \{X \in \mathcal{P}(\mathbb{N}) \mid X \text{ is finite}\}$$

For instance, the subset $\{2, 4, 6\} \in \mathcal{P}_{\mathcal{F}}(\mathbb{N})$ but the subset $\{1, 3, 5, \ldots\} \notin \mathcal{P}_{\mathcal{F}}(\mathbb{N})$. Show that $\mathcal{P}_{\mathcal{F}}(N)$ is countable.

Note: Here's an ordering that does **not** work: order the elements of $\mathcal{P}_{\mathcal{F}}(\mathbb{N})$ (finite subsets of $\mathbb{N}$) by their size (since each is finite, the size is well defined):

$$\phi, \{1\}, \{2\}, \{3\}, \{4\}, \ldots$$

Do you see the problem?

Hint: Every finite subset has a largest element.

Solution

$\mathcal{P}_{\mathcal{F}}(\mathbb{N})$ is countable. We list the subsets based on their largest element.

$$\emptyset, \{1\}, \{2\}, \{1,2\}, \{3\}, \{1,3\}, \{2,3\}, \{1,2,3\}, \{4\}, \ldots$$

There are 2^{i-1} sets that have i as the largest element. Therefore, the rank of any set that has i as its largest element is at most $1 + \sum j = 1^i 2^{j-1} = 1 + (2^0 + 2^1 + \ldots + 2^{i-1} = 1 + (2^i - 1) = 2^i$.

3. We will now show that $\mathcal{P}(\mathbb{N})$ is uncountable even though we already know this fact. This is an opportunity to practice the diagonal method. To do this, we will first represent each subset S of $\mathbb{N}$ by an infinite binary word in which the i^{th} bit is 1 if $i \in S$ and 0 otherwise. To practice this notion try to fill in the table:

infinite binary word	subset of $\mathbb{N}$
$00000\ldots$	$\emptyset$
$100000\ldots$	$\{1\}$
$011100000\ldots$	$\{2,3,4\}$
$010101000\ldots$	$\{2,4,6\}$
$1010101010\ldots$	$\{1,3,5,7,\ldots\}$
$1001001001\ldots$	$\{3k-2 \mid k \in \mathbb{N}\}$
$111111\ldots$	$\mathbb{N}$

Now we need to show that there is no bijection from $\mathbb{N}$ to the set of infinite binary words. Reproduce the diagonal argument to show this fact.

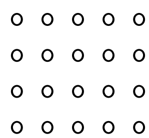
Note: Think about this, but you are not required to provide an answer: Why doesn't this diagonal argument disprove the fact that $\mathcal{P}_{\mathcal{F}}(\mathbb{N})$ of question 2 is countable?

Solution

Table filled above.

Given a bijection from $\mathbb{N}$ to the set of infinite binary words, we achieve a contradiction by constructing an infinite binary word x such that there is no $i \in \mathbb{N}$ with $f(i) = x$. This can be done by making the i^{th} bit of x different from the i^{th} bit of $f(i)$, simply by flipping bits.

4. Consider the following grid of 20 white dots



We color 9 dots black. Prove that three of the black dots make a line.

Solution

There are 9 black dots in 4 rows. By the pigeonhole principle, one row must contain $\lceil 9/4 \rceil = 3$ black dots. These dots are on a line.

5. How many numbers in $\{1, 2, \ldots, 546\}$ are not divisible by 2 **and** not divisible by 3 **and** not divisible by 7?

Hint: Negate the requirement, find the answer using inclusion-exclusion, then fix it to answer the original question.

Hint: Check your answer against $546 \times (1/2) \times (2/3) \times (6/7)$ (this does not always work by the way depending on the choice of numbers).

Solution

Let's count the number of integers divisible by 2 or 3 or 7. If we let S_2 be the set of integers divisible by 2, and similarly, define S_3 and S_7, then we want $|S_2 \cup S_3 \cup S_7|$. By the inclusion-exclusion principle, this is

$$|S_2| + |S_3| + |S_7| - |S_2 \cap S_3| - |S_2 \cap S_7| - |S_3 \cap S_7| + |S_2 \cap S_3| cap S_7$$

$$= \frac{546}{2} + \frac{546}{3} + \frac{546}{7} - \frac{546}{6} - \frac{546}{14} - \frac{546}{21} + \frac{546}{42}$$

$$= 273 + 182 + 78 - 91 - 39 - 26 + 13 = 390$$

Note that I did not use $\lfloor \ \rfloor$ above because all divisions are exact. Finally, the number we want is $546 - 390 = 156$.

6. Prove by induction that for all integers $n \geq 0$, $\sum_{i=0}^{n}(4i + 1) = 2n^2 + 3n + 1$.

Solution

Base case: When $n = 0$, we have $\sum_{i=0}^{0}(4i + 1) = 4 \cdot 0 + 1 = 1 = 2 \cdot 0^2 + 3 \cdot 0 + 1$

The inductive hypothesis is $P(k) : \sum_{i=0}^{k}(4i+1) = 2k^2+3k+1$.

Inductive step: We need to show the inductive step: $\forall k \geq 0, P(k) \Rightarrow P(k+1)$, where $P(k+1)$ is $\sum_{i=0}^{k+1}(4i+1) = 2(k+1)^2+3(k+1)+1$.

$$\sum_{i=0}^{k+1}(4i+1) = \sum_{i=0}^{k}(4i+1) + [4(k+1)+1] = 2k^2+3k+1+[4(k+1)+1]$$

$$= 2k^2+3k+1+(4k+4+1) = (2k^2+4k+2)+(3k+3)+1 = 2(k+1)^2+3(k+1)+1$$

Problem 1

A homogeneous subset of $\mathbb{N}$ is one where all the elements have the same parity. A student in CSCI 150 decided to prove that the set all homogeneous subsets of $\mathbb{N}$ is uncountable. The student proposed that every homogeneous subset of $\mathbb{N}$ can be represented as an infinite binary word with all its 1s either in even positions or in odd positions. For instance, $\{1, 3, 5, 7, 9 \ldots\}$ can be represented by the infinite binary word $1010101010\ldots\ldots$, and $\{2, 4, 10\}$ by $0101000001000\ldots$, and $\phi = \{\ \}$ by $000\ldots$, and so on.

With B being the set of all infinite binary words satisfying the above condition, the student mimicked Cantor's diagonalization proof in order to construct a infinite binary word w such that there is no $i \in \mathbb{N}$ with $f(i) = w$. He did this by flipping bits along the diagonal (as shown below).

(Hypothetical function $f : \mathbb{N} \to B$)

$\mathbb{N}$	B
1	$\underline{1}010101010\ldots$
2	$0\underline{1}01000001000\ldots$
3	$00\underline{0}\ldots$
$\vdots$	$\vdots$

$w = 001\ldots$

(a) What's wrong with the student's proof?

Solution

The constructed word w may not be in B. For instance, it might end up having two consecutive 1s, corresponding to one even and one odd integer in $\mathbb{N}$. Therefore, w may not correspond to a homogeneous subset of $\mathbb{N}$. So the proof is wrong.

(b) Fix the student's proof.

Solution

We can fix the proof by changing pairs of bits, as follows:

$$00 \to 01$$
$$01 \Rightarrow 00$$
$$10 \Rightarrow 00$$

This way, the constructed word w will be different from every word in B, but will only have 1s in even positions. Therefore, w will correspond to a homogenous set.

Problem 2

A 3 digit number is good iff it does **not** have 1 in the first digit, **and** does **not** have 2 in the second digit, **and** does **not** have 3 in the third digit. How many good 3 digit numbers are there?

(a) Solve this question using the product rule by identifying the possibilities for each of the three digits.

(b) Do the same using the inclusion-exclusion principle. *Hint*: consider the negation to get the "or" logic. In other words, find the number of bad 3 digit numbers, then adjust your answer to find the number of good 3 digit numbers.

(c) Assume now that all 3 digits must be different. How many 3 digit numbers are good? *Hint*: which technique is more suitable, that of part (a) or part (b)?

Solution

(a) $\underline{\#\text{ ways}}$

1. Choose the first digit. . . . . $8\ (\neq 0, \neq 1)$
2. Choose the second digit . . . . $9\ (\neq 2)$
3. Choose the third digit . . . . $\underline{9\ (\neq 3)}$

$8 \times 9 \times 9 = 648$

<u>There are 648 "good" 3-digit numbers.</u>

(b) Inclusion-Exclusion: counting "bad" 3-digit numbers.

1**: $10 \times 10 = 100$

2: $9 \times 10 = 90$

**3: $9 \times 10 = 90$

12*: 10

1*3: 10

*23: 9

123: 1

Number of bad 3-digit numbers $= 100 + 90 - 10 - 10 - 9 + 1 = 252$

Total number of 3-digit numbers: $9 \times 10 \times 10 = 900$
Number of good 3-digit numbers $= 900 - 252 = \underline{648}$

(c) Repeating the work in part (b):

1**: $9 \times 8 = 72$

2: $8 \times 8 = 64$

**3: $8 \times 8 = 64$

12*: 8

1*3: 8

*23: 7

123: 1

Number of bad 3-digit numbers $= 72 + 64 + 64 - 8 - 8 - 7 + 1 = 178$
Total number of 3-digit numbers: $9 \times 9 \times 8 = 648$
Number of good 3-digit numbers $= 648 - 178 = \underline{470}$

Problem 3

There is a $1 \times \ell$ rectangle. For simplicity, we will assume that ℓ is a positive integer.

(a) We place $\ell+1$ points in the rectangle. Prove that two of the points must be within a distance of $\sqrt{2}$.

Solution

This is a typical pigeonhole setting. Divide the rectangle into ℓ 1×1 squares. Placing $\ell+1$ points means that one square will have at least two points. The largest distance in the square is $\sqrt{2}$. Therefore, two of the points must be within a distance of $\sqrt{2}$.

(b) Show by an example that you can place $\ell+1$ points in the rectangle such that the distance between any two of them is at least $\sqrt{2}$.

Solution

The $\ell+1$ points will be placed on the perimeter of the rectangle. With the conceptual square divisions above, we start by placing the first point in the upper-left corner of the first square. We place the second point in the lower-left corner of the second square, and so on by alternating between upper-left and lower-left. The final point will end up in either the lower-right or upper-right corner of the last square.

It is easy to verify that the closest two points are a a distance of exactly $\sqrt{2}$.

(c) Show by an example that you can place ℓ points in the rectangle such that no two of them are within a distance of $\sqrt{2}$.

Solution

With the placement of points as described in part (b), move the second point horizontally to the right by ϵ. Move the third point in the same way by 2ϵ. Keep this pattern until the point before last is moved by $(\ell-1)\epsilon$ (se we need $(\ell-1)\epsilon \leq 1$). The last point is simply removed (to end up with ℓ points). Now, none of the points are within a distance of $\sqrt{2}$.