CSCI 150 Discrete Mathematics
Homework 8

Saad Mneimneh, Computer Science, Hunter College of CUNY

Solution

1. (This is somewhat piegonhole in reverse) A person wants to collect c identical coupons. There are n different types of coupons. How large must m be so that if this person collects m coupons, he/she is guaranteed to obtain c identical ones? (m must be a function of c and n).

   Solution

   We need $\lceil \frac{m}{n} \rceil = c$. The smallest m that works is $n(c-1)+1$.

2. My drawer has n pairs of socks (so $2n$ socks). The socks came back from the laundry, surprisingly with no sock missing! But they are all separated. Each sock is labeled either "left" or "right". Assume that I grab some socks blindly, how many do I need to guarantee a good pair?

   - If I must use a correct pair of socks (two that match). Show how to use the concept of pigeonhole to find the answer.
   - If I am happy to use any left sock with any right sock (this is not pigeonhole, but it has some logic to it).

   Solution

   Think of each of the n correct pairs of socks as a box.

- If we pick $n+1$ socks, 2 will be in the same box, so they belong to the same pair. Observe that n would not be enough since each of the n socks could end up in its own box.
- If we are happy with any "left-right" combination, we are again left with $n+1$ as the answer (we can't improve). Since there are n "left" socks and n "right" socks, $n+1$ socks can't all be the same orientation. Also observe that n would not be enough, since they can be all "left" or all "right."

3. Consider a binary word of length 70. Prove that there are at least two occurrences of some sequence of 6 bits.

   Solution

   In a binary word of length 70 there are 65 different starting indices for a sequence of 6 bits; however, there are only $2^6 = 64$ distinct 6 bit sequences. By the pigeonhole principle, $\lceil \frac{65}{64} \rceil = 2$; hence, there must be at least one repeat.

4. There is a contest with 40 Pokemons. There are 18 Pokemons who like to fight in the sky, and 23 who like to fight on ground. Several of them like to fight in the water. The number of those who like to fight in the sky and on ground in 9. There are 7 Pokemons who like to fight in the sky and in water, and 12 who like to fight on ground and in water. There are 4 Pokemons who like to fight in the sky, on ground, and in water. How many Pokemons like to fight in water?

   Solution

   We use the inclusion-exclusion principle to find the number of Pokemon who like to fight in water.

   $|S| = 18 \quad |G| = 23$
   $|S \cap G| = 9 \quad |S \cap W| = 7 \quad |G \cap W| = 12$
   $|S \cap G \cap W| = 4$

   Since we have 40 Pokemon in total, we have the following:

   $|S \cup G \cup W| = |S| + |G| + |W| - |S \cap G| - |S \cap W| - |G \cap W| + |S \cap G \cap W| = 40$

   Thus, $|W| = 23$

5. In a class of 20, 12 are boys and 13 wear glasses. There are twice as many boys with glasses as girls with no glasses. How many girls wear glasses?

Solution

Let's express all the information we have in terms of sets. We consider three sets: The set of boys B, the set of girls G, and the set of people with glasses L. We have the following information:

$$|B \cup G \cup G| = 20$$

$$|B| = 12$$

$$|L| = 13$$

$$|G| = 8 \text{ (deduced)}$$

$$|B \cap L| = 2(|G| - |G \cap L|) = 16 - 2|G \cap L|$$

$$|B \cap G| = 0$$

$$|B \cap G \cap L| = 0$$

The formula for inclusion-exclusion is:

$$|B \cup G \cup L| = |B| + |G| + |L| - |B \cap G| - |B \cap L| - |G \cap L| + |B \cap G \cap L|$$

$$20 = 12 + 8 + 13 - 0 - (16 - 2|G \cap L|) - |G \cap L| + 0$$

$$20 = 12 + 8 + 13 - 16 + |G \cap L|$$

So $|G \cap L| = 3$.

6. Consider the following well-known Fibonacci sequence given by

$$0, 1, 1, 2, 3, 5, 8, 13, 21, \ldots$$

where $F_0 = 0$ and $F_1 = 1$ and

$$F_n = F_{n-1} + F_{n-2}, \quad n \geq 2$$

Prove the following two statements by induction. One of the above requires strong induction and multiple base cases (which one and why?).

- For all $n \geq 0$, $\sum_{i=0}^{n} F_i = F_{n+2} - 1$

  Solution
  Base case is $n = 0$ and we have $\sum_{i=0}^{0} F_i = F_0 = 0 = 1-1 = F_2-1$.

  Inductive hypothesis, $P(k) : \sum_{i=0}^{k} F_i = F_{k+2} - 1$.

  Inductive step: $\forall k \geq 0, P(k) \Rightarrow P(k+1)$

  $$P(k+1) : \sum_{i=0}^{k+1} F_i = F_{k+3} - 1$$

  $$\sum_{i=0}^{k+1} F_i = \sum_{i=0}^{k} F_i + F_{k+1} = F_{k+2} - 1 + F_{k+1} = F_{k+2} + F_{k+1} - 1 = F_{k+3} - 1$$

  The proof requires $k \geq 0$ to apply the inductive hypothesis, so $n_0 = 0$ was enough for our base case.

- For all $n \geq 1$, $F_n \geq \phi^{n-2}$ where $\phi = (1+\sqrt{5})/2$. *Hint*: ϕ is the solution of a well-know quadratic equation.

  Solution
  Base cases: We will use the base cases for both $n = 1$ and $n = 2$. For the former, we have $F_1 = 1 \geq \frac{2}{1+\sqrt{5}} = \phi^{-1}$ where the inequality holds because $1 + \sqrt{5} > 2$. For the latter, we have $F_2 = 1 \geq 1 = \phi^0$.

  Inductive hypothesis: $\wedge_{1 \leq i \leq k} P(i)$: $F_i \geq \phi^{i-2}$ for all $1 \leq i \leq k$.

  Inductive step: $\forall k \geq 1, \wedge_{1 \leq i \leq k} P(k) \Rightarrow P(k+1)$.

  $$P(k+1) : F_{k+1} \geq \phi^{k-1}$$

  $$F_{k+1} = F_k + F_{k-1} \geq \phi^{k-2} + \phi^{k-3} = \phi^{k-1}$$

  as desired. The final equality comes from the following line of reasoning.

  $$\phi^{k-2} + \phi^{k-3} = \phi^{k-1}\Big(\frac{1}{\phi} + \frac{1}{\phi^2}\Big)$$

But we know that the quantity between parenthesis is 1 (see lecture notes on induction with Fibonacci).

The proof works as long as $k-1 \geq 0$ (so that F_{k-1} is defined), so we need $k \geq 1$ and our choice for $n_0 = 1$ is appropriate.

Problem
Assume that $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$.

(a) Suppose that we change the induction mechanism as follows:

- Base case: Prove that $P(0)$ is true
- Inductive step: Prove that for all $k \geq 0, P(k) \Rightarrow P(k+2)$

Explain why this would not constitute a valid proof that $P(n)$ is true for all $n \in \mathbb{N}$. How would you change the base case to obtain a valid proof?

Solution
This is not an appropriate mechanism if one wants to prove that the property is true for all $n \geq 0$. In particular, $P(1)$ is missed. This means that $P(n)$ for all odd n will be missed as well. To fix it, we can add another base case for $n = 1$. So make $P(1)$ as part of the base cases in addition to $P(0)$.
(b) Use the above strategy to prove the following statement:

$$\forall n \in \mathbb{N}, n^2 = 4m \vee n^2 = 4m + 1$$

where $k \in \mathbb{N}$.

Solution
Base cases:
$n = 0$, $0^2 = 4 \cdot 0$
$n = 1$, $1^2 = 4 \cdot 0 + 1$

Inductive hypothesis: $P(k) : k^2 = 4m \vee k^2 = 4m + 1$

Inductive step: $\forall k \geq 0$, $P(k) \Rightarrow P(k+2)$

$$P(k+2) : (k+2)^2 = 4m' \vee (k+2)^2 = 4m' + 1$$
$$(k+2)^2 = k^2 + 4 + 4k = k^2 + 4(k+1)$$

Therefore, $(k+2)^2 = 4m + 4(k+1) = 4(m+k+1) = 4m'$ or $(k+2)^2 = 4m + 1 + 4(k+1) = 4(m+k+1) + 1 = 4m' + 1$.

The proof works since

$$P(0) \Rightarrow P(2) \Rightarrow P(4) \Rightarrow \dots$$

$$P(1) \Rightarrow P(3) \Rightarrow P(5) \Rightarrow \dots$$

(c) Prove the same statement using a typical strong induction mechanism (it should not be very different from above).

Solution
Base cases:
$n = 0$, $0^2 = 4 \cdot 0$
$n = 1$, $1^2 = 4 \cdot 0 + 1$

Inductive hypothesis: $\wedge_{0 \le i \le k} P(i)$: $i^2 = 4m \vee i^2 = 4m + 1$ for all $0 \le i \le k$.

Inductive hypothesis: $\forall k \ge 1, \wedge_{0 \le i \le k} P(i) \Rightarrow P(k+1)$.

$$P(k+1) : (k+1)^2 = 4m' \vee (k+1)^2 = 4m' + 1$$

$$(k+1)^2 = [(k-1)+2]^2 = (k-1)^2 + 4 + 4(k-1) = (k-1)^2 + 4k$$

Therefore, either

$$(k+1)^2 = 4m + 4k = 4(m+k) = 4m'$$

or

$$(k+1)^2 = 4m + 1 + 4k = 4(m+k) + 1 = 4m' + 1$$

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The proof works as long as $0 \le k-1 \le k$, which means $k \ge 1$, so the choice of $n_0 = 1$ is appropriate.