CSCI 150 Discrete Mathematics
Homework 9

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Solution

1. An alien species communicates using a three-letter alphabet $\{x, y, z\}$. In their language, words must obey one rule: zz cannot be part of any word; otherwise, the speaker will go to sleep and never finish the sentence. Describe the number of words of length n by a recurrence. Let a_n be the number of words of length n, and express a_n in terms of a_{n-1} and a_{n-2}.

   Hint: Do as we did with the tiling problem, i.e. consider different cases based on how you start a word, then for each case figure out in how many ways you can finish it.

   Solution
   There are 4 possible ways a word in this language can start

$$x\ldots$$

$$y\ldots$$

$$zx\ldots$$

$$zy\ldots$$

   where "…" represents a word in the language of a smaller length.

Therefore, if we denote by a_n the number of words of length n, we have by the addition rule:

$$a_n = a_{n-1} + a_{n-1} + a_{n-2} + a_{n-2} = 2a_{n-1} + 2a_{n-2}$$

This recurrence can then be solved using the characteristic equation method.

2. Consider a version of the Tower of Hanoi where each disk is duplicated, so we have $2n$ disks with 2 disks of each size. The rules of the game are the same. Let a_n be the number of moves needed to solve this $2n-$disk problem.

(a) Find a recurrence for a_n.

Solution

We make two moves for each move of the original Tower of Hanoi. In other words, we always move the two copies of every disk together. This gives $a_n = a_{n-2} + 2 + a_{n-1} = 2a_{n-1} + 2$.

(b) Guess a solution for a_n is terms of n (by exploring), and prove it by induction.

Solution

We can easily guess that the number of moves must be exactly twice the number of moves in the original puzzle.

$$a_0 = 0, a_1 = 2, a_2 = 6, a_3 = 14, \ldots$$

Therefore, $a_n = 2(2^n - 1)$. Here's a proof by induction:

Base case: when $n = 0$, $a_0 = 0 = 2(2^0 - 1)$.

Inductive step: The inductive hypothesis is that $P(k) : a_k = 2(2^k - 1)$ is true. We want to prove that $\forall k \geq 0, P(k) \Rightarrow P(k+1)$. Using the recurrence:

$$a_{k+1} = 2a_{k-1} + 2 = 2[2(2^k - 1)] + 2 = 2^{k+2} - 4 + 2 = 2^{k+2} - 2 = 2(2^{k+1} - 1)$$

3. Consider a sequence where $a_0 = 1$, $a_1 = -2$, and $a_n = -2a_{n-1} - a_{n-2}$ for $n \geq 2$.

(a) Guess a_n as a function of n and prove it by strong induction.

Solution

$$a_0 = 1, a_1 = -2, a_3 = 4, a_4 = -5, \ldots$$

We guess that $a_n = (n+1)(-1)^n$. To prove it:

Base case:

$$a_0 = (0+1)(-1)^0 = 1$$
$$a_1 = (1+1)(-1)^1 = -2$$

Inductive step: The inductive hypothesis is that $P(k) : a_i = (i+1)(-1)^i$ is true for all $0 \leq i \leq k$. We want to prove that $\forall k \geq 1, \wedge_{0 \leq i \leq k} P(i) \Rightarrow P(k+1)$. Consider a_{k+1}:

$$a_{k+1} = -2a_k - a_{k-1} = -2(k+1)(-1)^k - [(k-1)+1](-1)^{k-1} = 2(k+1)(-1)^k - k(-1)^{k-1}$$

$$= 2(k+1)(-1)^{k+1} - k(-1)^{k+1} = (2k+2-k)(-1)^{k+1} = [(k+1)+1](-1)^{k+1}$$

The proof works as long as $k - 1 \geq 0$, so that a_{k-1} is defined. Therefore, the base case stopping at $n_0 = 1$ is enough.

(b) Use the characteristic equation method.

We have $x^2 = -2x - 1$ which is equivalent to $x^2 + 2x + 1 = 0$ and $(x+1)^2 = 0$, which has two solutions that are the same $p = q = -1$. Therefore, $a_n = c_1(-1)^n + c_2 n(-1)^n$. Using a_0 and a_1, we find that $c_1 = c_2 = 1$ and get $a_n = (n+1)(-1)^n$.

4. Consider the following recurrence,

$$a_n = \frac{1}{2}a_{n-1} + 1$$

where $a_1 = 1$.

(a) Guess a pattern for a_n and prove it by induction.

Solution

$a_1 = 1$

$a_2 = \frac{1}{2} + 1 = \frac{3}{2}$

$a_3 = \frac{1}{2}\frac{3}{2} + 1 = \frac{3}{4} + 1 = \frac{7}{4}$

$a_4 = \frac{1}{2}\frac{7}{4} + 1 = \frac{7}{8} + 1 = \frac{15}{8}$

$\vdots$

$a_n = \frac{2^n - 1}{2^{n-1}} = 2 - \frac{1}{2^{n-1}}$

Here's a proof by induction:

Base case: $a_1 = 2 - \frac{1}{2^{1-1}} = 2 - \frac{1}{2^0} = 2 - 1 = 1$

Inductive step: The inductive hypothesis is that $P(k) : a_k = 2 - \frac{1}{2^{k-1}}$ is true, and we want to prove that $\forall k \geq 1, P(k) \Rightarrow P(k+1)$. Consider a_{k+1} and use the recurrence:

$$a_{k+1} = \frac{1}{2}a_k + 1 = \frac{1}{2}\left[2 - \frac{1}{2^{k-1}}\right] + 1 = 1 - \frac{1}{2^k} + 1 = 2 - \frac{1}{2^{(k+1)-1}}$$

(b) Convert the recurrence for a_n into the form $a_n = Aa_{n-1} + Ba_{n-2}$ by eliminating the constant 1 in the recurrence. Solve for a_n using the characteristic equation.

Solution

$$a_n = \frac{1}{2}a_{n-1} + 1$$
$$a_{n-1} = \frac{1}{2}a_{n-2} + 1$$

Therefore, $a_n - a_{n-1} = \frac{1}{2}a_{n-1} - \frac{1}{2}a_{n-2}$.

$$a_n = \frac{3}{2}a_{n-1} - \frac{1}{2}a_{n-2}$$

The characteristic equation is $x^2 - \frac{3}{2}x + \frac{1}{2} = 0$, so consider the equation

$$2x^2 - 3x + 1 = 0$$

which has the two solutions $p = 1$ and $q = 1/2$. Therefore,

$$a_n = c_1(1)^n + c_2\left(\frac{1}{2}\right)^n$$

$a_1 = c_1 + c_2/2 = 1$ $a_2 = c_1 + c_2/4 = 3/2$ This gives $c_1 = 2$ and $c_2 = -2$, resulting in $a_n = 2 - 2\frac{1}{2^n} = 2 - \frac{1}{2^{n-1}}$.

(c) Find a_n using the generating function method (follow the example illustrated in class).

Solution

$f(x) = a_1x + a_2x^2 + a_3x^3 + \ldots$

$f(x) = a_1x + (\frac{1}{2}a_1 + 1)x^2 + (\frac{1}{2}(a_2 + 1)x^3 + \ldots$

$f(x) = (a_1x + x^2 + x^3 + \ldots) + \frac{1}{2}(a_1x^2 + a_2x^3 + \ldots)$

$f(x) = x(1 + x^2 + x^3 + \ldots) + \frac{x}{2}(a_1x + a_2x^2 + \ldots)$

$f(x) = \frac{x}{1-x} + \frac{x}{2}f(x)$

$f(x)[1 - \frac{x}{2}] = \frac{x}{1-x} \Rightarrow f(x) = \frac{x}{(1-x)(1-\frac{x}{2})} = \frac{A}{1-x} + \frac{B}{1-\frac{x}{2}}$

Matching the numerator $A(1 - x/2) + B(1 - x)$ with x, we find that $-A/2 - B = 1$ and $A + B = 0$, so $A = 2$ and $B = -2$. Therefore,

$$f(x) = \frac{2}{1-x} - \frac{2}{1-x/2}$$

$$= 2(1 + x + x^2 + \ldots) - 2(1 + \frac{x}{2} + \frac{x^2}{4} + \ldots)$$

So the coefficient of x^n is $a_n = 2 - 2\frac{1}{2^n} = 2 - \frac{1}{2^{n-1}}$.

5. Consider $2n$ points on the circumference of a circle. In how many ways can we join the points pairwise by n chords such that no two chords intersect? Call this number a_n, find a recurrence for it, then solve it.

Solution

Pick any of the $2n$ points on the circumference of the circle. By joining this point to another one on the circle, we must end up with an even number of points on each side of the segment. This number can be $0, 2, 4, ..., 2n - 2$. We can then recursively do the same thing for each side. Therefore, if a_n is the number of ways we can join the $2n$ points, we must have

$$a_n = a_0 a_{n-1} + a_1 a_{n-2} + \dots a_{n-1} a_0$$

Since $a_0 = 1$ (with no points there is only one way), this is exactly the Calatan number recurrence. Therefore, $a_n = \frac{1}{n+1}\binom{2n}{n}$.

Problem 1 (optional)
Consider the following sequences starting at $a_0, a_1, \ldots$:

$$5, -10, 20, -40, \ldots$$

$$1, 7, 49, 343, \ldots$$

(a) For each of the sequences above, find a recurrence of the form $a_n = Aa_{n-1}$ for $n \geq 1$, and solve for a_n as a function of n.

Solution
The recurrences are: $a_n = -2a_{n-1}$ with $a_0 = 5$, and $a_n = 7a_{n-1}$ with $a_0 = 1$, respectively. For the first recurrence, we get $a_n = 5(-2)^n$, and for the second recurrence, we get $a_n = 7^n$.

(b) For each of the sequences above, find a recurrence of the form $a_n = Aa_{n-1} + Ba_{n-2}$ for $n \geq 2$, by considering the recurrence from part (a) for a_n and a_{n-1}; the solution is not unique, depending on how you combine recurrences, so find the solution that corresponds to adding up the recurrences.

Solution
Consider the first recurrence:

$$a_n = -2a_{n-1}$$

$$a_{n-1} = -2a_{n-2}$$

So we get: $a_n + a_{n-1} = -2a_{n-1} - 2a_{n-2}$. Therefore, $a_n = -3a_{n-1} - 2a_{n-2}$, $n \geq 2$. Similarly, for the second recurrence, we have $a_n = 6a_{n-1} + 7a_{n-2}$, $n \geq 2$. Note: Solving using the characteristic equation will retrieve the exact same solutions as in (a).

(c) There are infinitely many recurrences of the form $a_n = Aa_{n-1} + Ba_{n-2}$ that work since we can write $a_n = cp^n + 0 \cdot q^n$ for $q \neq p$. Find a recurrence of the form $a_n = Aa_{n-1} + Ba_{n-2}$ for $n \geq 2$ that works for both sequences at the same time.

Solution
We need $p = -2$ and $q = 7$ to be the two solutions of the characteristic equation. Therefore, we need

$$(x+2)(x-7) = 0$$

This gives $x^2 = 5x + 14$. Therefore, the recurrence $a_n = 5a_{n-1} + 14a_{n-2}$ will work for both.

Problem 2 (optional)
Consider the following recurrence:

$$a_n = a_{n-1} - a_{n-2}$$

where $a_0 = 0$ and $a_1 = 1$.

(a) Using the recurrence and the initial conditions, generate the first 18 numbers of the sequence $\{a_n\}$. Try to guess a way to compute a_n immediately by simply knowing n.

Solution
Here are the first few terms in the sequence:

$$0, 1, 1, 0, -1, -1, 0, 1, 1, 0, -1, -1, \ldots$$

This is a repetition of the pattern $0, 1, 1, 0, -1, -1$. Therefore, we can say the following:

$$a_n = \begin{cases} 0 & a_n \equiv 0, 3 \pmod{6} \\ 1 & a_n \equiv 1, 2 \pmod{6} \\ -1 & a_n \equiv 4, 5 \pmod{6} \end{cases}$$

(b) Solve for a_n. *Hint*: observe that a_n has the form $a_n = Aa_{n-1} + Ba_{n-2}$.

Solution
When we form the characteristic equation, we get $x^2 - x + 1 = 0$, which has the following solutions:

$$\frac{1 + i\sqrt{3}}{2}$$

So $a_n = c_1\left(\frac{1+i\sqrt{3}}{2}\right)^n + c_2\left(\frac{1-i\sqrt{3}}{2}\right)^n$. Solving for c_1 and c_2 using a_0 and a_1, we get $c_1 = -c_2 = c = -i\frac{\sqrt{3}}{3}$.

(c) Your expression for a_n in part (b) will contain the imaginary number i. Use the binomial theorem to obtain a nicer expression for a_n:

$$a_n = \frac{1}{2^{n-1}}\left[\binom{n}{1}3^0 - \binom{n}{3}3^1 + \binom{n}{5}3^2 - \ldots\right]$$

Solution

$$a_n = \frac{c}{2^n}[(1+i\sqrt{3})^n - (1-i\sqrt{3})^n]$$

$$a_n = \frac{c}{2^n}\Big[\sum_k = 0^n \binom{n}{k}(i\sqrt{3})^k - \sum_{k=0}^{n}\binom{n}{k}(-i\sqrt{3})^k\Big]$$

The even power will cancel each other, so we get:

$$a_n = \frac{c}{2^n}\Big[\sum_{k \text{ odd}}\binom{n}{k}(i\sqrt{3})^k - \sum_{k \text{ odd}}\binom{n}{k}(-i\sqrt{3})^k\Big]$$

$$a_n = \frac{c}{2^n}\Big[\sum_{k \text{ odd}}\binom{n}{k}(i\sqrt{3})^k + \sum_{k \text{ odd}}\binom{n}{k}(i\sqrt{3})^k\Big]$$

$$a_n = \frac{c}{2^{n-1}}\Big[\sum_{k \text{ odd}}\binom{n}{k}(i\sqrt{3})^k\Big]$$

$$a_n = \frac{-i\sqrt{3}}{3\cdot 2^{n-1}}\Big[\sum_{k \text{ odd}}\binom{n}{k}(i\sqrt{3})^k\Big]$$

$$a_n = \frac{-1}{3\cdot 2^{n-1}}\Big[\sum_{k \text{ odd}}\binom{n}{k}(i\sqrt{3})^{k+1}\Big]$$

$$a_n = \frac{-1}{2^{n-1}}\Big[\sum_{k \text{ odd}}\binom{n}{k}i^{k+1}\sqrt{3}^{k-1}\Big]$$

The even powers of i will provide the alternating signs, and the even power of $\sqrt{3}$ will provide the powers of 3.